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What are quantum measurable spaces?

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The formalism of Hilbert spaces as a mathematical foundation for quantum mechanics has become standard, indicating that the expressiveness of the abstract theory is sufficient to capture the essential features of quantum phenomena. From an algebraic perspective, structures like C^*-algebras[1] play an important role in this story, serving as algebras of observables. Due to the inherent probabilistic nature of quantum theory, one often speaks of *quantum probability* in this context.

Conceptually, one may expect classical probability to be *precisely* the special case of quantum probability where the observable algebras under consideration are commutative. Indeed, the commutativity condition enforces classical behavior: the Heisenberg uncertainty principle no longer applies, and all observables can meaningfully be assigned joint values. A first step in this direction is provided by the well-known ***Gelfand duality***, which establishes a contravariant equivalence of categories[2] between compact Hausdorff spaces with continuous maps and commutative unital C^*-algebras with unital $*$-homomomorphisms.

From a probabilistic perspective, however, $*$-homomomorphisms are too restrictive, and one instead considers ***completely positive unital maps*** (cpu maps). These are linear maps $\phi\colon \mathcal{A} \to \mathcal{B}$ satisfying two important properties:

- *Unitality*: $\phi(1) = 1$, which corresponds to normalization of probability measures (a probability measure μ on X satisfies $\mu(X) = 1$);
- *Positivity*: $\phi(a) \geq 0$ for all $a \geq 0$, which corresponds to nonnegativity of probability measures ($\mu(A) \geq 0$ for all measurable sets $A \subseteq X$);

  The condition of being *completely* positive means that positivity holds also for the associated matricial versions $M_n(\phi)\colon M_n(\mathcal{A}) \to M_n(\mathcal{B})$, where $M_n(\phi)((a_{i,j})_{i,j}) \coloneqq (\phi(a_{i,j}))_{i,j}$.

Similarly, in probability theory one generalizes probability measures to ***Markov kernels***: Informally, these are parametrized families of probability measures, and they naturally arise in practice as regular conditional probabilities. Explicitly, a Markov kernel between measurable spaces (X, Σ_X) and (Y, Σ_Y) is a map $f(- \mid =)\colon \Sigma_Y \times X \to [0,1]$ such that $f(- \mid x)$ is a probability measure on Y for every $x \in X$ and $f(A \mid =)$ is a measurable function for every $A \in \Sigma_Y$.

The ***probabilistic Gelfand duality*** [FJ15] shows that cpu maps between commutative C^*-algebras correspond to *continuous* Markov kernels between their associated spaces. For finite-dimensional studies, this result is sufficient to content oneself with C^*-algebras and cpu maps. However, for infinite-dimensional cases, the topological nature of these results is unsatisfactory. First, our criterion that the commutative case of quantum probability should precisely be classical measure-theoretic probability does not hold, since in the latter context, Markov kernels are only required to be measurable, not continuous. Second, those Markov kernels that arise in mathematical practice (e.g., as regular conditional probabilities) are not continuous in general.

These reasons make us think that the category of C^*-algebras is not the best setting for quantum probability theory. Further evidence comes from quantum measurement theory. We recall that a ***positive operator-valued measure*** (POVM) on a measurable space (X, Σ_X) is a map $\mu\colon \Sigma_X \to$

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[1] In the present abstract, C^*-algebras and homomorphisms between them are always unital, and we usually leave this implicit.

[2] We roughly recall that a category is a collection of objects and morphisms between them, where morphisms can be composed and each object has an identity morphism. A contravariant equivalence between two categories is a correspondence that reverses the direction of morphisms, induces bijections between the sets of parallel morphisms, and matches objects of the two categories up to isomorphism.

$\mathcal{B}(\mathcal{H})$, where $\mathcal{B}(\mathcal{H})$ is the C^*-algebra of bounded operators over the Hilbert space $\mathcal{H}$, satisfying the following:

- *Normalization*: $\mu(X) = 1$;
- *Positivity*: $\mu(A) \geq 0$ for all $A \in \Sigma_X$;
- *σ-additivity*: for every countable sequence $(A_n)_{n\in\mathbb{N}}$ of pairwise disjoint measurable sets, we have $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$.

On physical and operational grounds, one would expect that a morphism of observable algebras maps a POVM on one algebra to a POVM on the other algebra. But if every $*$-homomorphism, or even every cpu map, is a morphism, then this is not guaranteed: the image of a POVM may fail the σ-additivity required of a POVM.

It is often suggested that a good setting for quantum probability theory is that of W^*-algebras (von Neumann algebras) instead. This is indeed well-motivated by the idea that W^*-algebras can be thought of as algebras of noncommutative random variables modulo almost everywhere equality, and there is a Gelfand-type duality that makes this precise [Pav22]. But from our point of view, this is not satisfactory either, because quotienting by almost everywhere equality means that one needs to fix a reference measure. In the non-atomic case, one thereby loses access to one of the most basic concepts of probability theory, namely delta measures.

For these reason, we investigate what ***quantum measurable spaces*** should be. We draw inspiration from the class of C^*-algebras

$$\mathcal{L}^\infty(X) := \{f\colon X \to \mathbb{C} \text{ bounded measurable}\},$$

where (X, Σ_X) is a measurable space. In contrast to standard C^*-algebras, where only uniform limits are guaranteed, in $\mathcal{L}^\infty(X)$ pointwise limits also exist. These limits can be described using the monotone convergence theorem, which states that for any bounded monotone increasing sequence of measurable functions, the pointwise supremum is also measurable.

Analogously, we say that a C^*-algebra $\mathcal{A}$ is a ***σC^*-algebra*** (also known as a *monotone σ-complete C^*-algebra*) if for every norm-bounded monotone increasing sequence

$$a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots \leq \|b\|,$$

there exists a supremum $\sup_n a_n$ with respect to $\leq$. Accordingly, a cpu map (resp., a $*$-homomorphism) is **σ-cpu** (resp., a σ-homomorphism) if $\phi(\sup_n a_n) = \sup_n \phi(a_n)$ for every norm-bounded monotone increasing sequence $(a_n)_{n\in\mathbb{N}}$. Until now, σC^*-algebras have been largely overlooked, although they do appear in the literature — most notably in parts of the books by Saito and Wright [SW15] and Pedersen [Ped18] — as an intermediate structure between C^*-algebras and W^*-algebras.

Our main result with respect to homomorphisms is the following.

Theorem 1 (Measurable Gelfand duality [FL26, Theorem 4.2.5]). *There is an idempotent adjunction*[3]

$$\begin{tikzcd} \left\{\begin{array}{c}\textit{Measurable spaces with}\\ \textit{measurable functions}\end{array}\right\} \arrow[r, bend left, "\mathcal{L}^\infty"] \arrow[r, phantom, "\perp"] & \left\{\begin{array}{c}\textit{Commutative } \sigma C^*\textit{-algebras}\\ \textit{with } \sigma\textit{-homomorphisms}\end{array}\right\}^{\mathrm{op}} \arrow[l, bend left] \end{tikzcd}$$

The fact that one only obtains an adjunction seems unsatisfactory, and below we will address this problem by restricting to meaningful (full) subcategories.

For the moment, let us explain via examples why we do not recover a full equivalence. First, there are non-isomorphic measurable spaces that become isomorphic when one considers the associated C^*-algebras. For a simple example, consider the set with two elements $\{0, 1\}$ equipped with the power set σ-algebra $\mathcal{P}(\{0, 1\})$ and the measurable space X given by the real numbers $\mathbb{R}$ equipped with the σ-algebra generated by a proper nonempty set E, which corresponds to $\{\varnothing, E, \mathbb{R} \setminus E, \mathbb{R}\}$. These two measurable spaces are not isomorphic, but they give rise to the same C^*-algebra $\mathbb{C}^2$.

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[3] For the purposes of this abstract, this can be thought of as a weaker version of an equivalence.

Second, not all commutative σC^*-algebras are isomorphic to some $\mathcal{L}^\infty(X)$. An important example is given by the von Neumann algebra $L^\infty([0,1])$, whose elements are equivalence classes of bounded measurable functions with respect to the Lebesgue measure.[4] This can be seen by considering σ-states, i.e. σ-cpu maps with codomain $\mathbb{C}$. The σC^*-algebra $\mathcal{L}^\infty(X)$ admits *pure* σ-states[5] given by evaluation at points $x \in X$. In contrast, $L^\infty([0,1])$ has no pure σ-states, because all of its σ-states are normal states ([FL26, Example 3.2.12]), and it is well-known that $L^\infty([0,1])$ has no pure normal states. In other words, $\mathcal{L}^\infty(X)$ allows integration against delta measures, while $L^\infty([0,1])$ does not.

We now focus on restricting the measurable Gelfand duality to a contravariant equivalence. To this aim, we first consider what a meaningful restriction would be. In the commutative setting, recent work by Jamneshan and Tao [JT23] argues that a good choice for the objects is given by ***Baire measurable spaces***. These spaces extend *standard Borel spaces*, the go-to framework for most of probability theory, beyond the countable setting in a natural way. For instance, regular conditional probabilities still exist [Fre03, Corollary 452N].

In our formalism, we can ensure that Baire measurable spaces correspond to a "σ-version of the double dual": given a C^*-algebra $\mathcal{A}$, one considers its closure under taking suprema of norm-bounded monotone increasing sequences in its enveloping von Neumann algebra $\mathcal{A}^{**}$. This yields the σC^*-algebra $\mathcal{A}^\infty$, called the ***Pedersen–Baire envelope*** of $\mathcal{A}$.

Theorem 2 (Pedersen–Baire duality [FL26, Corollary 4.2.9])**.** *Measurable Gelfand duality restricts to a contravariant equivalence*

$$\left\{\begin{matrix}\textit{Baire measurable spaces with}\\ \textit{measurable functions}\end{matrix}\right\} \xrightarrow[\simeq]{\mathcal{L}^\infty} \left\{\begin{matrix}\textit{Commutative Pedersen–Baire envelopes}\\ \textit{with } \sigma\textit{-homomorphisms}\end{matrix}\right\}^{\mathrm{op}}$$

Moreover, standard Borel spaces correspond to commutative Pedersen–Baire envelopes of separable *C^*-algebras.*

Turning to probabilistic morphisms, we note that the concept of a POVM extends naturally by allowing an arbitrary σC^*-algebra as codomain in place of $\mathcal{B}(\mathcal{H})$. In particular, for every measurable space (X, Σ_X), there is a canonical POVM $\chi_-\colon \Sigma_X \to \mathcal{L}^\infty(X)$ given by sending each measurable subset $A \in \Sigma_X$ to its characteristic function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

The following theorem was already proven in [Wri72, Section 2]; we generalize it to abstract (Boolean) σ-algebras in [FL26, Theorem 4.1.30].

Theorem 3. *Let (X, Σ_X) be a measurable space and $\mathcal{A}$ a σC^*-algebra. Then for every POVM $\mu\colon \Sigma_X \to \mathcal{A}$, there is a unique σ-cpu map $\phi\colon \mathcal{L}^\infty(X) \to \mathcal{A}$ which extends μ, in the sense that the diagram*

$$\begin{tikzcd} \Sigma_X \arrow[rr, "\chi_-"] \arrow[dr, "\mu"'] & & \mathcal{L}^\infty(X) \arrow[dl, "\phi"] \\ & \mathcal{A} & \end{tikzcd}$$

commutes.

In particular, σ-cpu maps preserve POVMs by direct check, and therefore one obtains a bijective correspondence between POVMs $\mu\colon \Sigma_X \to \mathcal{A}$ and σ-cpu maps $\mathcal{L}^\infty(X) \to \mathcal{A}$. Moreover, by setting $\mathcal{A} = \mathcal{L}^\infty(Y)$, Markov kernels from Y to X correspond to POVMs $\Sigma_X \to \mathcal{L}^\infty(Y)$. These two observations yield a fully faithful contravariant functor[6] from the category of measurable spaces with Markov kernels to the category of σC^*-algebras with σ-cpu maps. By restricting to Baire measurable spaces, we obtain the following.

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[4] Concretely, $f, g\colon [0,1] \to \mathbb{C}$ are equivalent if $\mu_L(\{x \in [0,1] \mid f(x) = g(x)\}) = 1$, where μ_L denotes the Lebesgue measure.

[5] A pure σ-states is a σ-state that is extremal in the convex set of all σ-states.

[6] Roughly, a fully faithful contravariant functor is an inclusion that restricts to a contravariant equivalence.

Theorem 4 ([FL26, Corollary 4.3.8]). *There is a contravariant equivalence*

$$\left\{\begin{matrix}\textit{Baire measurable spaces}\\ \textit{with Markov kernels}\end{matrix}\right\} \xrightarrow[\simeq]{\mathcal{L}^\infty} \left\{\begin{matrix}\textit{Commutative Pedersen–Baire envelopes}\\ \textit{with } \sigma\textit{-cpu maps}\end{matrix}\right\}^{\mathrm{op}}$$

Moreover, standard Borel spaces correspond to commutative Pedersen–Baire envelopes of separable *C^*-algebras.*

Overall, this investigation motivates some preliminary definitions for quantum measurable spaces. In particular, one may define ***quantum Baire measurable spaces*** as Pedersen–Baire envelopes, and ***quantum standard Borel spaces*** as the Pedersen–Baire envelopes of separable C^*-algebras. In addition to the well-behaved nature of their commutative side and the preservation of POVMs, these candidate quantum measurable spaces also have crucial features: functional calculus generalizes to this setting ([FL26, Section 3.4]), and pure σ-states suffice to test equality between positive elements ([FL26, Theorem 3.3.21]; see also [Ped18, Corollary 4.5.13]).

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