BV-Categories of Spacetime Interventions: Extended Abstract
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We use the Chu construction to functorially build BV-categories from duoidal categories, demonstrating that candidate models of BV-logic can be cofreely constructed from a fragment of a model of Guglielmi's sequencing operator. By using this construction to show that the strong Hyland envelope is a BV-category, we find a way to build a canonical model of spatio-temporal relationships between agents in spacetime from any symmetric monoidal category. The concrete physical interpretation of spacetime events in this model as intervention-context pairs resolves deficiencies in previous attempts to give a general categorical semantics to quantum supermaps - in particular, those which are unitary.

1 Introduction

Monoidal category theory provides a formalisation of circuit theories, that is, theories of processes with timelike and spacelike composition rules [9]. However, whilst monoidal structure captures a minimal notion of spacetime compatibility of a theory, it is not rich enough to provide a full account of higher-order quantum computation [22] or equivalently of the causal structure and correlations between spacetime events as conceptualised in the modern foundations of physics [7, 16, 18].

In this domain, spatio-temporal correlations are studied by modelling events in terms of the interventions that can be made by agents (see the left of figure 1). Such agents possess the ability to perform experiments by choosing the settings and viewing the outcomes of devices in their local laboratories [16]. This formulation of events in terms of the local interventions of agents motivates a dual picture of spacetime as a *process-with-holes*, referred to variously as the supermap [5, 4, 6], process matrix [16], or process tensor perspective [18]. These spacetime models are innately higher-order: a process-with-holes is interpreted as a context to be probed by the lower-order interventions of the local agents, thereby acting as a map on the processes of some monoidal category (see the right of figure 1).

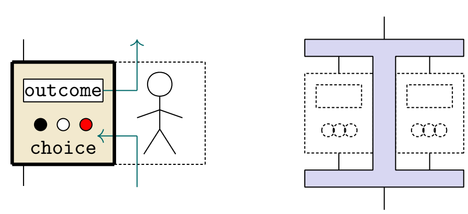

Figure 1: Left: An intervention, Right: A context with two holes.

With the understanding that a fully-fledged model of spatio-temporal correlations requires us to move from processes to spacetime events, we expect that we should move away from mere monoidal categories

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to a more capable logical system. The deep inference system BV is one such logic originating in the study of extensions of multiplicative linear logic (MLL) to incorporate non-commutativity [20, 19, 12, 13]. BV-logic is defined in terms of the *calculus of structures* and adds a non-commutative sequencing operator $\varolessthan$ to the multiplicative conjunction $\otimes$ and disjunction $\parr$ of MLL [12]. These structures permit the graphical depiction given in figure 2.

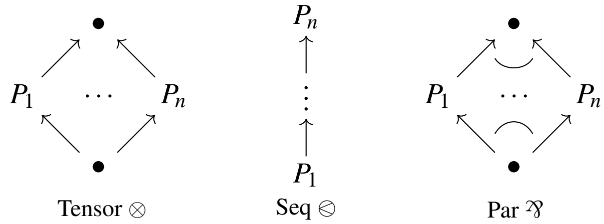


Figure 2: Graphs for the calculus of structures [12].

The similarity between these structures and spacetime diagrams is particularly notable. Indeed, the analogy between the operators $\otimes$, $\varolessthan$ and $\parr$ and spatio-temporal correlations was appreciated in the seminal [12]. Upon interpreting the P_i as the agents, we see that $\otimes$ forbids communication, $\parr$ allows arbitrary communication and $\varolessthan$ permits just one directional communication. Indeed, Guglielmi suggests a broad interpretation of $\varolessthan$ as temporal composition with $P_1 \varolessthan P_2$ thought of as "P_1 before P_2".

2 Technical Contributions

The main technical contribution of the linked preprint is to categorify a result of [1] to prove that the Chu construction [8] is a right 2-adjoint to the forgetful functor U from BV-categories to $\otimes$-closed, $\otimes$-symmetric normal duoidal categories, lifting the well-known 2-adjunction between closed symmetric monoidal categories and $*$-autonomous ones [17].

$$\mathsf{CSNDuo} \underset{U}{\overset{\mathsf{Chu}}{\underset{\longleftarrow}{\overset{\longrightarrow}{\top}}}} \mathsf{BV}$$

This allows us to construct cofree BV-categories over the fragment given by multiplicative conjunction $\otimes$, sequencing $\varolessthan$ and a weak dualising operator $(-)^* = [-, \bot]$. While it is already very well-known that $\mathsf{Chu}(\mathscr{C}, \bot)$ is a model of linear logic, the interesting part here is that the tensor product $\varolessthan$ can be lifted to give a sequencing operator on $\mathsf{Chu}(\mathscr{C}, \bot)$ with the required self-duality and distributive structures with respect to the tensor $\otimes$ and par $\parr$. Along the way, we reframe the definition of BV-categories to show that they are pseudomonoids in a 2-category of $*$-autonomous categories placing them on a firmer algebraic footing. This allows us to define the 2-category BV of BV-categories, BV-functors and BV-transformations and handle all the coherence data in a concise manner.

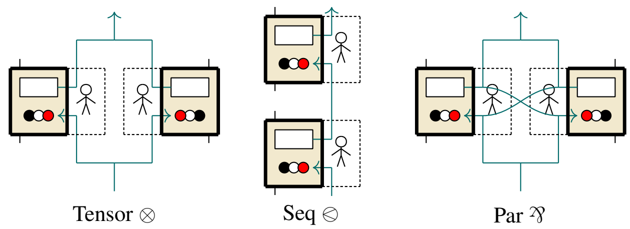
With this tool at hand, we build upon the work of [14] by studying a modified version of the Hyland envelope [15, 21] which we dub the *strong* Hyland envelope. This allows us to construct BV-categories of higher-order processes over arbitrary symmetric monoidal categories, completing a series of canonical free and cofree constructions starting from any symmetric monoidal category $\mathscr{C}$ to produce a BV-category $\mathsf{StEnv}(\mathscr{C})$.

$$\mathscr{C} \overset{D}{\rightsquigarrow} \mathscr{C} \times \mathscr{C}^{\mathrm{op}} \overset{\mathscr{P}}{\rightsquigarrow} \mathsf{Prof}(\mathscr{C}) \overset{N}{\rightsquigarrow} \mathsf{StProf}(\mathscr{C}) \overset{\mathsf{Chu}}{\rightsquigarrow} \mathsf{StEnv}(\mathscr{C})$$

Here, D is the doubling (splicing) construction of [10]; $\mathscr{P}$ is the free cocompletion; N is the normalisation of duoidal categories [11, 10]; and Chu is the Chu construction [8, 2]. Besides $\otimes$ which is lifted all the way along this series of adjunctions, all the other logical connectives ($\olessthan$, $\parr$ and $(-)^*$) alongside their necessary interaction, are (co)freely added.

3 Application to higher-order quantum operations

Intuitively, objects of $\mathsf{StEnv}(\mathscr{C})$ consist of intervention-context pairs. This unites the two dual views of spacetime events: an event is no longer just the local actions of an agent, nor the hole in a process, but a unification of the two. The causal relationships between these events are captured by the tensors $\otimes$, $\olessthan$ and $\parr$ which act on the interventions in the following fashion.



These tensors come equipped with distributors which act as,

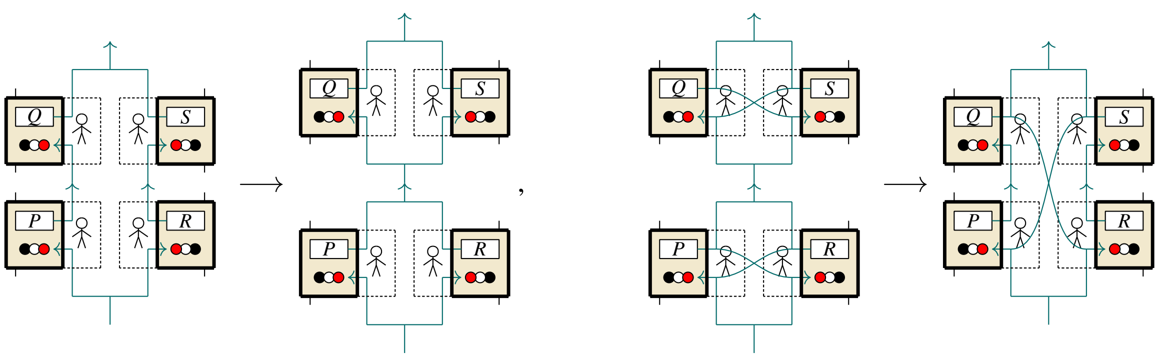


Figure 3: The distributors $(P \olessthan Q) \otimes (R \olessthan S) \to (P \otimes R) \olessthan (Q \otimes S)$ and $(P \parr R) \olessthan (Q \parr S) \to (P \olessthan Q) \parr (R \olessthan S)$.

This allows us to build a model of spacetime events supporting the previous tensor products from *any* underlying theory of first-order processes.

Concretely, $\mathsf{StEnv}(\mathscr{C})$ is not just a canonical and physically well-motivated BV-category, but one which recovers those already used in the literature to model higher-order quantum operations. First, thanks to the results of [14], this model recovers higher-order quantum theory (the theory of multi-input superchannels with and without definite causal order) when applied to the theory of quantum channels. Furthermore, when applied to unitary channels, it recovers the unitary superchannels [3, 23, 24] and generalises them to incorporate sequenced and iterated higher-order types. Beyond recovering and extending both mixed and pure higher-order quantum theory, the approach builds models of higher-order processes from *any* lower-order theory: for example, BV models for higher-order operations on infinite-dimensional systems.

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