Higher order maps in operational probabilistic theories

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1 Introduction

Sequential and concurrent models of classical or quantum computation, such as Turing machines and circuits, describe computation as sequences of state changes over time. In contrast, models like the lambda calculus involve higher-order operations on functions, corresponding physically to transformations acting on other transformations. This perspective underlies Higher-Order Quantum Theory (see Ref. [1] for a review), where quantum channels themselves can be inputs to "second-order" maps, giving rise to a hierarchy of higher-order maps (HOMs). Within this hierarchy, an important subclass is given by combs[2, 3], which can be realized as quantum circuits with open slots that are filled with compatible maps. This framework has been widely used to optimize information-processing tasks involving objects more general than quantum states, such as channel discrimination, tomography, cloning, and storage and retrieval of transformations. However, not all higher-order maps admit such a realization. A key example is the quantum switch[4], which coherently superposes different sequential orders of operations. Such maps exhibit indefinite causal order, a feature that can provide advantages over standard circuit-based approaches in various tasks[5, 6].

In this submission, we first review the axiomatic framework for higher-order quantum maps [7], analogous to Kraus' axiomatization of quantum operations, in which we characterize the admissible higher-order maps, i.e., those compatible with the probabilistic structure of quantum theory. Using the Choi–Jamiolkowski isomorphism, the admissible higher-order maps correspond to positive operators satisfying a set of linear constraints. We then address the compositional structure of higher-order maps[8], namely the ways in which the quantum systems of different higher-order maps can be connected to produce new ones. We prove that the admissibility of a composition requires verifying a sequence of no-signaling conditions, thereby linking a computational property of a map (its functional description) to a fundamental physical property of the process (its causal structure). Finally, we present a generalization of the higher-order framework to general physical theories described by the formalism of Operational Probabilistic Theories (OPTs)[9]. Our approach enriches an OPT with the notion of a contraction, a transformation resembling a categorical trace that extends the compositional structure beyond purely sequential composition and enables the definition of HOM hierarchies. We prove that a (convex) OPT admits such a contraction if and only if it is endowed with Choi states and Choi effects. Within this framework, the higher-order functionality of an OPT is therefore not an extrinsic addition, but is fully determined by its intrinsic compositional structure.

2 Axiomatic approach to higher order maps

The axiomatic approach to higher-order quantum theory begins by considering maps from quantum transformations to quantum transformations and recursively constructing a hierarchy of higher-order maps. The construction relies on the notion of *type*, which, similarly to a functional description, specifies what is the input and what is the output of a higher order map. The type x of a map R identifies the position of a map within the hierarchy. Types are defined recursively:

Definition 1 (Types)**.** *Every finite dimensional quantum system corresponds to an* elementary type A. *The elementary type corresponding to the tensor product of quantum systems A and B is denoted with AB. The type of the trivial system is denoted by I. We denote with* $\mathsf{EleTypes}$ *the set of elementary types. Let* $\mathsf{A} := \mathsf{EleTypes} \cup \{(\} \cup \{)\} \cup \{\rightarrow\}$

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be an alphabet. We define the set of types *as the smallest subset* $\mathsf{Types} \subset \mathsf{A}^*$ *such that*[1]

$$\mathsf{EleTypes} \subset \mathsf{Types}, \quad x, y \in \mathsf{Types} \implies (x \to y) \in \mathsf{Types} \tag{1}$$

The idea is that a map transforming maps of type x into maps of type y has type $(x \to y)$. Thus, a type x can be represented as a string, e.g. $x = (((A_1 \to A_2) \to (A_3 \to A_4)) \to A_5)$. We denote by I the trivial one-dimensional system. Higher-order maps must also satisfy minimal admissibility requirements ensuring compatibility with the probabilistic structure of quantum theory. Admissible maps must be linear, as they must respect convex combinations. Using the Choi–Jamiolkowski isomorphism, a map of type x is represented by an operator in $\mathscr{L}(\mathscr{H}_x)$, where $\mathscr{H}_x := \bigotimes_{i \in x} \mathscr{H}_i$, and i runs over the non-trivial elementary types appearing in x. For example, if M has type $((A \to B) \to I) \to C$, then $M \in \mathscr{L}(\mathscr{H}_x)$ and $\mathscr{H}_x = \mathscr{H}_A \otimes \mathscr{H}_B \otimes \mathscr{H}_C$. Formally we give the following definition:

Definition 2 (Generalized higher order map)**.** *If x is a type in* Types*, the set of* generalized maps of type x*, denoted by* $\mathsf{T}_{\mathbb{R}}(x)$*, is defined by the following recursive definition.*

- *if A is an elementary type, then every $M \in \mathscr{L}(\mathscr{H}_A)$ is a generalized map of type A, i.e.* $\mathsf{T}_{\mathbb{R}}(A) := \mathscr{L}(\mathscr{H}_A)$.
- *if x, y are two types, then every Choi operator of linear maps $\mathscr{M} : \mathsf{T}_{\mathbb{R}}(x) \to \mathsf{T}_{\mathbb{R}}(y)$, is a generalized map M of type $(x \to y)$.*

Besides linearity, another requirement is that an admissible higher-order map of type $x \to y$ must map admissible maps of type x to admissible maps of type y. This is analogous to the condition that a quantum operation maps physical states (i.e., positive matrices with trace smaller or equal then one) to physical states. However, for quantum operations one imposes a stronger requirement: they must preserve physicality even when applied locally to a bipartite system (complete positivity). To generalize this idea to the hierarchy of higher-order maps, we introduce the notions of extension of a type x by an elementary type E, and of extended higher-order maps.

Definition 3 (Extension with an elementary type and extended higher order map)**.** *Let x be a type, E be an elementary type and $M \in \mathsf{T}_{\mathbb{R}}(x)$. The* extension $x \parallel E$ *of x by the elementary type E is defined recursively as follows:*

$$\textit{for } A, E \in \mathsf{EleTypes} \textit{ we have } A \parallel E := AE; \quad \textit{for } x, y \in \mathsf{Types} \textit{ we have } (x \to y) \parallel E := (x \to y \parallel E). \tag{2}$$

We denote with M_E the extension of M by E *which is defined recursively as follows: If x, y are two types and $M \in \mathsf{T}_{\mathbb{R}}(x \to y)$ then $M_E \in \mathsf{T}_{\mathbb{R}}(x \parallel E \to y \parallel E)$ is the Choi operator of the map $\mathscr{M} \otimes \mathscr{I}_E : \mathsf{T}_{\mathbb{R}}(x \parallel E) \to \mathsf{T}_{\mathbb{R}}(y \parallel E)$, where $\mathscr{I}_E : \mathscr{L}(\mathscr{H}_E) \to \mathscr{L}(\mathscr{H}_E)$ is the identity map.*

The notion of extended map enables us to define admissible maps for all types in the hierarchy. This definition is recursive. We begin by specifying that deterministic maps of elementary type correspond to normalized states, i.e., positive operators with unit trace. Admissible maps of elementary type, on the other hand, are subnormalized states, namely positive operators with trace less than or equal to one. The admissibility condition for the entire hierarchy is then defined as follows.

Definition 4 (Admissible higher order map)**.** *Let $x, y \in$* Types *be two types, $M \in \mathsf{T}_{\mathbb{R}}(x \to y)$ be an generalized map of type $x \to y$ and $M_E \in \mathsf{T}_{\mathbb{R}}(x \parallel E \to y \parallel E)$ be the extension of M by E. Let $\mathscr{M} : \mathsf{T}_{\mathbb{R}}(x) \to \mathsf{T}_{\mathbb{R}}(y)$ and $\mathscr{M} \otimes \mathscr{I}_E : \mathsf{T}_{\mathbb{R}}(x \parallel E) \to \mathsf{T}_{\mathbb{R}}(y \parallel E)$ be the linear maps whose Choi operator are M and M_E respectively.*

We say that M is admissible

1. *for all elementary types E, the map sends admissible maps of type $x \parallel E$ to admissible maps of type $y \parallel E$.*
2. *there exist $\{N_i\}_{i=1}^n \subseteq \mathsf{T}_{\mathbb{R}}(x \to y)$, $0 \leq n < \infty$ such that, for all elementary types E,*
   - *$\forall 1 \leq i \leq n$ the map $\mathscr{N}_i$ satisfies item 1,*
   - *For all elementary types E, the map $(\mathscr{M} + \sum_{i=1}^n \mathscr{N}_i) \otimes \mathscr{I}_E$ maps admissible maps of type $x \parallel E$ to admissible maps of type $y \parallel E$ and deterministic maps of type $x \parallel E$ to deterministic maps of type $y \parallel E$*

The set of admissible maps of type $x \to y$ is denoted with $\mathsf{T}(x \to y)$. An operator $D \in \mathsf{T}_{\mathbb{R}}(x \to y)$ is a deterministic *map of type $x \to y$, if $D \in \mathsf{T}(x \to y)$ and $(\mathscr{D} \otimes \mathscr{I}_E)$ maps deterministic admissible maps of type $x \parallel E$ to deterministic admissible maps of type $y \parallel E$. The set of deterministic maps of type x is denoted as $\mathsf{T}_1(x)$*

[1] Please note that A^* stands for the set of words of the alphabet A

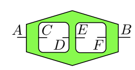
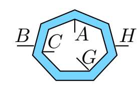
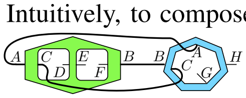
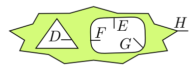
Notice that we have introduced the notion of *deterministic maps*, together with the requirement that any admissible map must belong to a collection of admissible maps whose sum is a deterministic map. This condition encodes a normalization constraint, ensuring that probabilities greater than one never arise within the framework. Definition 4 generalises Kraus' axiomatic definition of quantum operations to higher-order maps. Indeed, one can easily verify that, for the simplest case $x = A \to B$, definition 4 reduces to the notion of completely positive trace non increasing map. Moreover, this axiomatic approach does not explicitly rely on the mathematical properties of maps in the hierarchy. The structure of quantum theory enters only at the ground level—elementary types—and propagates inductively throughout the hierarchy. The mathematical characterization of admissible higher-order maps is given in the following proposition.

Proposition 1. *Let x be a type and $M \in \mathscr{L}(\mathscr{H}_x)$ a map of type x. Let $\mathsf{Hrm}(\mathscr{H})$ and $\mathsf{Trl}(\mathscr{H})$ denotes the subspace of hermitian operator and traceless hermitian operators, respectively. Then M is* admissible *if and only if $M \geq 0$ and $M \leq D$ for a deterministic map D. A map D of type x is* deterministic *if and only if $D \geq 0$ and*

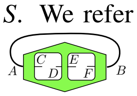
$$D = \lambda_x I_x + X_x, \quad \lambda_x \geq 0, \quad X_x \in \Delta_x \subseteq \mathsf{Trl}(\mathscr{H}_x) \tag{3}$$

where λ_x and Δ_x are defined recursively as $\lambda_E = \frac{1}{d_E}$, $\lambda_{x\to y} = \frac{\lambda_y}{d_x \lambda_x}$, $\Delta_E = \mathsf{Trl}(\mathscr{H}_E)$, $\Delta_{x\to y} = [\mathsf{Hrm}(\mathscr{H}_x) \otimes \Delta_y] \oplus [\overline{\Delta}_x \otimes \Delta_y^\perp]$, where $\Delta^\perp$ denotes the orthogonal complement (with respect to the Hilbert-Schmidt inner product) of Δ in $\mathsf{Hrm}(\mathscr{H})$ while $\overline{\Delta}$ is the orthogonal complement in $\mathsf{Trl}(\mathscr{H})$.

2.1 Composing higher order maps

A quantum channel is a map of type $A \to B$ and it is supposed to take as an input a state of system (type) A and to output a state of system B. However, this is not the only way we are allowed to use a channel. For example we can compose a channel from A to B with a channel from B to C and create a channel from A to C. What about composing higher order maps? Let's consider two maps $R =$  of type x and $T =$  of type y. Intuitively, to compose R and T should mean to connect the elementary systems that they share, i.e. $R * T :=$  $=$  . In terms of the Choi isomorphism, this operation is given by the so called *link product*, namely $R * T = \mathrm{Tr}_{\mathsf{S}}[(R \otimes I_{\mathsf{B}\setminus\mathsf{S}})(I_{\mathsf{A}\setminus\mathsf{S}} \otimes T^{\theta_{\mathsf{S}}})]$,where $R \in \mathscr{L}(\mathscr{H}_{\mathsf{A}})$, $T \in \mathscr{L}(\mathscr{H}_{\mathsf{B}})$, $\mathsf{S} := \mathsf{A} \cup \mathsf{B}$ is the set of systems that R and S share and $T^{\theta_{\mathsf{S}}}$ is the partial traspose of T on the systems of S. However, not every composition between two arbitrary maps of type x and y can be expected to be physically admissible. At the very least, we require that a composition rule for types x and y be admissible only if, for any maps of those types, their composition yields an object that can be interpreted as an admissible higher-order map. This motivates the following definition:

Definition 5. *Let x and y be two types such that they share a set of elementary types. We say that the composition $x * y$ is* admissible *if $\forall R \in \mathsf{T}_1(x), \forall T \in \mathsf{T}_1(y), \exists z$ s.t. $R * T \in \mathsf{T}_1(z)$*

The composition of two higher-order maps R and S amounts to connecting some systems of R with some systems of S. We refer to the elementary operation of connecting two systems of a map as a *contraction*, e.g. $\mathscr{C}_{AB}(R) =$  . The admissibility of the composition of two types can then be recast in terms of the admissibility of a set of such contractions, by chaining together the verification of admissibility for each individual contraction in a sequence.

We will prove that a contraction between systems A and B is admissible if no information flows from A to B. Before doing so, we need the following lemma, which shows that all higher-order maps can be interpreted (and used) as multipartite quantum channels.

Lemma 1. *For any type x, the set of elementary types appearing in its expression can be partitioned into two subsets, denoted in_x and out_x, such that any deterministic map of type x can be interpreted as a channel from in_x to out_x. This partition can be computed directly from the expression of the type.*

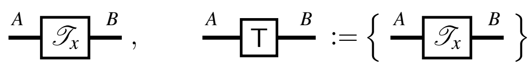
We are now ready to state the following proposition

Proposition 2. *Let x be a type, $A \in \mathsf{in}_x$, and $B \in \mathsf{out}_x$. Then we have that $\mathscr{C}_{AB}(x)$ is admissible $\iff A \not\mapsto_x B$ where $A \not\mapsto_x B$ means that any map R of type x, when considered as channel from in_x to out_x is no signalling from A to B.*

The previous proposition establishes a connection between the causal and compositional structure of a type: admissible contractions are precisely those compatible with the underlying causal structure. Consequently, the admissibility of a set of contractions—and hence of a composition—amounts to verifying a sequence of no-signaling conditions.

3 Operational Probabilistic Theories (OPTs)

In this section, we schematically review the framework of operational probabilistic theories [10]. The basic primitives concept in of the OPT framework are the notions of *test*, *system*, *events* and *probabilities*. A test T represents the idealization of an experimental procedure, carried on by some agent, which takes as input a physical system A and produces a physical system B as output. Each test consists of a finite set of events, $\mathsf{T} = \{\mathscr{T}_x\}$ and the interpretation is the following. If the agent performs the experimental procedure T results in an outcome x, then the system undergoes the transformation $\mathscr{T}_x$. It is helpful to represent these notion diagrammatically. Systems are depicted as wires, while tests and events are illustrated as boxes, as shown below:



(4)

Tests and events are equipped with notions of parallel (denoted with $\boxtimes$) and sequential composition (denoted with $\circ$), representing the concurrent and sequential execution of experimental procedures, respectively. We also consider the possibility of exchanging systems, requiring that the order in the parallel composition of tests is irrelevant up to a permutation of the systems. The language of category theory allows for a concise description of the outlined framework:

Definition 6 (category of operational tests)**.** *A category of operational tests is a symmetric strict monoidal category $\mathscr{T}$ whose objects are called systems, $\mathsf{ob}(\mathscr{T}) := \mathsf{Sys}$, and whose morphism are called tests, $\mathsf{mor}(\mathscr{T}) := \mathsf{Test}$, such that:*

1. *each test T is a finite*[2] *set of elements called events, i.e. $\mathsf{T} = \{\mathscr{T}_x\}$;*
2. *the collection Ev of all the events and the collection of sytems Sys form a symmetric strict monoidal category $\mathscr{E}$ where $\mathsf{ob}(\mathscr{E}) := \mathsf{Sys}$ and $\mathsf{mor}(\mathscr{E}) := \mathsf{Ev}$;*
3. *the sequential and parallel composition of $\mathscr{T}$ and $\mathscr{E}$ obey $\mathsf{S} \circ \mathsf{T} = \{\mathscr{S}_x \circ \mathscr{T}_y\}$ and $\mathsf{S} \boxtimes \mathsf{T} = \{\mathscr{S}_x \boxtimes \mathscr{T}_y\}$.*

We refer to the system I, corresponding to the neutral element of the parallel composition, as the trivial system. *For a pair of physical system A and B we denote with $\mathsf{Test}(A \to B)$ the set of test from A and B and with $\mathsf{Ev}(A \to B)$ the set of events from A and B, also called* transformations. *In particular, events of the kind $\mathsf{Ev}(I \to A)$ are called* states , *events of the kind $\mathsf{Ev}(A \to I)$ are called* effects *and events of the kind $\mathsf{Ev}(I \to I)$ are called* scalars. *We will not draw the wire corresponding to the trivial system.*

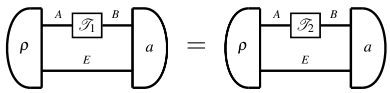
We then equip the category of operational tests with a *coarse-graining* map "$\curlyvee$", which captures the idea that an agent can discard part of the information about the outcome of an experiment. The coarse graining map must be commutative, associative and be compatible with parallel and sequantial composition (we omit the detail).

Up to this point, we have introduced a descriptive framework specifying which physical tests can be performed and which events are physically possible. However, a physical theory should also provide quantitative predictions about the occurrence of events associated with a given experiment. This is achieved by endowing the operational language with a probabilistic structure, namely a rule that assigns probabilities to the scalar events of the theory. What we get is called an operational probabilistic theory:

Definition 7 (Operational probabilistic theory)**.** *An* operational probabilistic theory Ω *is a operational category of test with a coarse graining map such that:*

[2] the generalization to the infinite case is possible

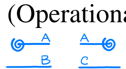
1. *The scalar events are probabilities and the scalar tests are probability distributions,* 0 *is a scalar event and the sequential composition of scalar events is the multiplication of real numbers.*
2.

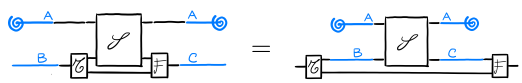
$$\mathscr{T}_1 = \mathscr{T}_2 \iff$$  $$\forall \rho \in \mathsf{Ev}(I \to AE),\ \forall \alpha \in \mathsf{Ev}(BE \to I). \qquad (5)$$

The meaning of Equation (5) is that two events yielding the same probabilities for all possible experiments are operationally indistinguishable and must therefore be identified.

4 Higher order maps in OPTs

In analogy with approaches that reconstruct quantum theory from information-theoretic axioms, we promote key theorems of higher-order quantum theory to the status of postulates. Let us recall two such facts concerning the compositional structure of higher-order *quantum* maps that we outlined in Section 2.1: (i) contraction (i.e., connecting systems in a loop) is the primitive operation underlying the composition of higher-order maps, and (ii) every higher-order map can be regarded as a channel. We then take these two features as foundational postulates for constructing a higher-order theory on top of an OPT. Firstly, we introduce the notion of a contraction:

Definition 8 (Operational contraction)**.** *Let* Ω *be an OPT and let* $A \in \mathsf{Sys}$*. An* operational contraction *for* A *is a map* $C_A :=$  $: \mathsf{CEv}(AB \to AC) \to \mathsf{Ev}(B \to C)$*, such that:*

- $C_A(\mathscr{E} \curlyvee \mathscr{D}) = C_A(\mathscr{E}) \curlyvee C_A(\mathscr{D})$.
-  *(locality).*
-  *(Yank equation).*

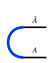
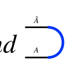
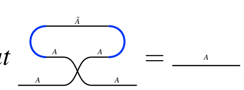
where $\mathsf{CEv}(AB \to AC)$ *is a suitable some collection of events closed under coarse graining.*

From a categorical perspective, a contraction can be understood as a weaker notion of a trace [11]. With this tool we can define a notion of an admissible higher order map in an OPT framework

Definition 9. *An admissible higher order map of type* $x \to y$ *is an event* $\mathscr{R}$ *from system* in_x *to systems* out_x *such that* $C_x(\mathscr{R} \boxtimes \mathscr{S})$ *is an admissible map of type* y*.*

Notice that this construction avoids the need to define extended maps and extended types, since higher-order maps are assumed from the outset to be events of the OPT. In the language of quantum theory, this corresponds to assuming from the beginning that a higher-order map is a quantum operation. Our construction relies on the notion of contraction and does not explicitly invoke the existence of a Choi–Jamiołkowski isomorphism. However, it is possible to prove that if an OPT is convex and admits contractions, then it also admits a Choi state and a Choi effect.

Proposition 3. *Let* Ω *be a convex OPT. Then the following are equivalents:*

1. *The theory has a contraction for system A*
2. *there exist a system* $\tilde{A} \in \mathsf{Sys}$*,*  $\in \mathsf{Ev}_{\mathbb{R}}(I \to \tilde{A}A)$*, and*  $\in \mathsf{Ev}_{\mathbb{R}}(\tilde{A}A \to I)$ *such that* .

The second condition is referred to as the existence of a Choi state and a Choi effect, as it allows to establishe a bijection between (generalized) transformations and (generalized) bipartite states, analogous to the Choi–Jamiołkowski isomorphism. This can be viewed as a weaker notion of compact closedness [12]. For convex theories, the construction based on contraction then reduces to analogous constructions for compact closed theories [13] (see also [14]). It is also natural to compare this result with the embedding of traced categories into compact closed ones via the geometry of interaction [15]. In this sense, one may say that a traced category (in the sense of admitting a contraction) that is also an OPT (with Eq. (5) playing a crucial role is compact closed (in the sense of admitting Choi states and effects).

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