Denotational semantics for stabiliser quantum programs (extended abstract)

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The problem of compiling quantum algorithms into fault-tolerant hardware-level instructions is a central challenge in the design of scalable quantum systems [6, 2, 16]. To this end, quantum error-correcting codes play a central role, where stabiliser codes are the most common and well-studied [9]. For fault-tolerant compilation to scale, we need a better understanding of the compositional structure of fault-tolerance, and therefore of the stabiliser fragment. Unlike general quantum programs, stabiliser quantum programs can be simulated efficiently on a probabilistic classical computer [1]. Despite this fact, the formal denotational semantics of stabiliser quantum programs has not been thoroughly studied, although their unitary evolution is well understood [17].

In this article, we develop a nondeterministic denotational semantics for quantum programs built from stabiliser operations, including Clifford operators, Pauli errors, Pauli measurement and classically-controlled Pauli operators. Our finely tuned denotational semantics for stabiliser quantum programs is to be contrasted with the usual, much larger denotational semantics of non-stabilizer quantum programs in terms of quantum channels between finite dimensional C^*-algebras. Our work draws from two lines of research: the categorical semantics of quantum programming languages and quantum computing [20, 19, 18, 13]; and the symplectic representation of pure stabiliser circuits [11, 15, 12, 8, 3]. Ultimately, these results constitue the first step towards the development of formally verified fault-tolerant quantum compilation frameworks.

1 Background: Semantics for mixed quantum theory

The categorical semantics of quantum theory builds on the mathematical semantics of finite-dimensional quantum processes with measurement and classical control. These semantics can be formally stated in the language of operator algebras [4], and are built in three stages of increasing expressivity:

(1) *Pure quantum mechanics* via finite-dimensional Hilbert spaces;
(2) *Mixed quantum mechanics* via completely-positive maps between matrix algebras;
(3) *Quantum measurements and classical control* via completely-positive maps between finite-dimensional C^*-algebras.

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These increasing stages of expressivity can be restated by applying the following functorial constructions to the †-compact-closed category, FHilb, of finite-dimensional Hilbert spaces and linear maps:

$$\textit{pure}\ \mathrm{QM} \xrightarrow{\text{CPM construction [18]}} \textit{mixed}\ \mathrm{QM} \xrightarrow{\text{Splitting †-idempotents [19]}} \mathrm{QM\ w/}\ \textit{measurements}$$

Finite-dimensional quantum mechanics can therefore be understood in purely categorical terms, agnostic to the theory of operator algebras. This point of view is highly amenable to generalisation and specialisation simply replace FHilb with any other †-compact-closed category, and apply these constructions to add abstract notions of mixing and measurement.

2 The symplectic representation of stabiliser quantum processes

In this article, we work with †-compact-closed categories specifically tailored to the stabiliser fragment. The first semantics is obtained directly by restricting FHilb to the stabiliser fragment. In the other hand, the second semantics is based on the symplectic representation of stabiliser codes. This second semantics is therefore naturally adapted for reasoning about QEC properties important for fault-tolerant compilation.

2.1 The symplectic representation of stabiliser codes

It is well-understood [11, 10] that Pauli operators on n-qupit quantum systems can be represented by elements of $\mathbb{Z}_p^n$:

$$\pi : \mathbb{Z}_p^n \oplus \mathbb{Z}_p^n \longrightarrow \mathcal{P}_p^{\otimes n} : (\mathbf{x}, \mathbf{z}) \longmapsto \exp\big(\pi i/p \cdot \mathbf{x}^\top \mathbf{z}\big) \bigoplus_{k=1}^{n} X_k^{x_k} Z_k^{z_k}, \tag{1}$$

where X, Z are the Pauli operators on the Hilbert space $\mathrm{span}\{|x\rangle \mid x \in \mathbb{Z}_p\}$.

Similarly stabiliser codes Π_S are in bijection with *isotropic* subspaces $S \subseteq \mathbb{Z}_p^{2n}$; that is to say the subspaces where *all elements commute* with respect to the symplectic form: $S \subseteq S^\omega$. The theory of stabiliser quantum error correction can be framed entirely in terms of S, rather than Π_S. By interpreting Pauli operators $\pi(\mathbf{e})$ as errors, we have:

An error $\mathbf{e} \in \mathbb{Z}_p^{2n}$ *is*	*Symplectic condition*	*Projector condition*
Trivial	$\mathbf{e} \in S$	$\Pi_S \pi(\mathbf{e}) \Pi_S = \Pi_S$
Detectable	$\mathbf{e} \notin S^\omega$	$\Pi_S \pi(\mathbf{e}) \Pi_S = 0$
Undetectable, nontrivial	$\mathbf{e} \in S^\omega \setminus S$	$\Pi_S \pi(\mathbf{e}) \Pi_S \neq 0,\ \Pi_S \pi(\mathbf{e}) \Pi_S \neq \Pi_S$

Moreover, a finite set $\mathcal{E} \subseteq \mathbb{Z}_p^{2n}$ of errors is **correctable** if and only if:

$$\forall \mathbf{e} \neq \mathbf{f} \in \mathcal{E} : \mathbf{f} - \mathbf{e} \notin S^\omega \quad \Longleftrightarrow \quad \forall \mathbf{e} \neq \mathbf{f} \in \mathcal{E} : \Pi_S \pi(\mathbf{f} - \mathbf{e}) \Pi_S = 0.$$

The **code distance** $d(S) \in \mathbb{N}$ of a nonempty affine coisotropic subspace $S = L + \mathbf{a}$ is the minimal number of tensor factors on which a nontrivial undetectable Pauli acts. This is most easily understood in the symplectic picture where:

$$d(S) \coloneqq \min \Big\{ \big|\{i \in \{0, \dots, n-1\} : (e_{x,i}, e_{z,i}) \neq (0,0)\}\big| \ : \ \forall \mathbf{e} = (\mathbf{e}_x, \mathbf{e}_z) \in S^\omega \setminus S \Big\}$$

2.2 Relational semantics for stabiliser processes

We give two semantics for stabiliser quantum processes. The first semantics is obtained directly by restricting FHilb. The other semantics is given by extending the symplectic representation to stabiliser quantum processes, and then composing the corresponding subspaces as relations. By applying the same categorical constructions as before, we have the following correspondences between both pictures:

$$\begin{array}{ccc}
\text{pure stabiliser theory} & \xleftrightarrow{\text{[15, Chapter 9],[8]}} & \text{affine Lagrangian relations} \\
\downarrow{\scriptstyle \text{CPM construction}} & & \downarrow{\scriptstyle \text{CPM construction}} \\
\text{mixed stabiliser theory} & \xleftrightarrow{\simeq} & \text{affine coisotropic relations} \\
\downarrow{\scriptstyle \text{Splitting } \dagger\text{-idempotents}} & & \downarrow{\scriptstyle \text{Splitting } \dagger\text{-idempotents}} \\
\begin{array}{c}\text{stabiliser theory}\\ \text{with measurements}\end{array} & \xleftrightarrow{\simeq} & \begin{array}{c}\text{affine relations}\\ \text{with symplectic types}\end{array}
\end{array}$$

Whilst the full symplectic semantics breaks down for qubits, we nevertheless recover the theory of CSS codes [5, 21, 7, 14].

In the final relational semantics (with measurement), quantum datatypes are captured by a symplectic modality Q. This enforces that quantum systems carry symplectic structure whereas classical ones do not: quantum systems are represented as $2n$-dimensional symplectic $\mathbb{Z}_p$-vector spaces $Q(\mathbb{Z}_p^{2n})$ and classical systems are n-dimensional $\mathbb{Z}_p$-vector spaces $\mathbb{Z}_p^n$.

These two semantics allow one to describe post-selected measurements and the resulting classical control. We also recover a characterisation of those stabiliser processes that are physically realisable, i.e. completely positive and trace-preserving:

Theorem 2.1 (Informal). *The physically-realisable stabiliser operations, including Pauli measurements, correspond to the* total *relations, where quantum data are captured by a* symplectic *modality.*

3 A small imperative programming language for QEC

This makes it possible to give semantics to a simple imperative language SPL (Stabiliser Programming Language) for stabiliser quantum programs, including Pauli measurement and classically-controlled Pauli operators. The terms are generated from the following grammar with respect to some linearly ordered set indexing registers:

$$c, d \ ::= \ c \,\text{;}\, d \mid \textbf{init}\ \underline{x} \mid \underline{\mathbf{y}} = A * \underline{\mathbf{x}} \mid \textbf{disc}\ \underline{x} \mid \textbf{qinit}\ \underline{x} \mid \underline{\mathbf{x}} *= U \mid \textbf{meas}\ \underline{x} \mid \textbf{ctrl}_P\ \underline{x}\ \underline{y} \mid \textbf{skip}$$

The term $c \,\text{;}\, d$ represents the sequential composition of subterms; **init** $\underline{x}$ represents the initialisation of $\underline{x}$ as the p-ary digit 0; $\underline{\mathbf{y}} = A * \underline{\mathbf{x}}$ applies the affine transformation A to $\underline{\mathbf{x}}$ and stores the result on $\underline{\mathbf{y}}$; **disc** $\underline{x}$ takes the trace of $\underline{x}$; **qinit** $\underline{x}$ represents initialisation of $\underline{x}$ as the qu$p$it $|0\rangle$; $\underline{\mathbf{x}} *= U$ applies the Clifford operator U on $\underline{\mathbf{x}}$; **meas** $\underline{x}$ represents the Pauli-X measurement on $\underline{x}$; $\textbf{ctrl}_P\ \underline{x}\ \underline{y}$ applies the Pauli operator P on $\underline{y}$, classically controlled by $\underline{x}$; and **skip** represents the identity.

Term	**Interpretation**	**Type**
$[\![c \,;\, d]\!]$	$\{\, (u,w) \;:\; \exists v:\; (u,v) \in [\![c]\!] \text{ and } (v,w) \in [\![d]\!] \,\}$	if $[\![c]\!] : A \to B$ and $[\![d]\!] : B \to C$ then $[\![c \,;\, d]\!] : A \to C$
$[\![\mathbf{init}]\!]$	$\{\, (*, a) \;:\; a = 0 \in \mathbb{Z}_p \,\}$	$\{*\} \to \mathbb{Z}_p$
$[\![\mathbf{qinit}]\!]$	$\{\, (*, (x,z)) \;:\; x = 0,\; z \in \mathbb{Z}_p \,\}$	$\{*\} \to Q(\mathbb{Z}_p^2)$
$[\![A*]\!]$	$\{\, (\mathbf{a}, \mathbf{b}) \;:\; \mathbf{b} = M\mathbf{a} + \mathbf{c} \,\}$	$\mathbb{Z}_p^{\oplus n} \oplus \mathbb{Z}_p^{\oplus m}$
$[\![\mathbf{meas}]\!]$	$\{\, ((x,z), a) \;:\; a = x \in \mathbb{Z}_p \,\}$	$Q(\mathbb{Z}_p^2) \to \mathbb{Z}_p$
$[\![\mathbf{skip}]\!]$	$\{\, (*, *) \,\}$	$\{*\} \to \{*\}$
$[\![\mathbf{disc}]\!]$	$\{\, ((x,z), *) \;:\; x, z \in \mathbb{Z}_p \,\}$	$Q(\mathbb{Z}_p^2) \to \{*\}$
$[\![\mathbf{ctrl}_X]\!]$	$\{\, ((a,x,z), (a, x+a, z)) \;:\; a, x, z \in \mathbb{Z}_p \,\}$	$\mathbb{Z}_p \oplus Q(\mathbb{Z}_p^2) \to \mathbb{Z}_p \oplus Q(\mathbb{Z}_p^2)$
$[\![\mathbf{ctrl}_Z]\!]$	$\{\, ((a,x,z), (a, x, z+a)) \;:\; a, x, z \in \mathbb{Z}_p \,\}$	$\mathbb{Z}_p \oplus Q(\mathbb{Z}_p^2) \to \mathbb{Z}_p \oplus Q(\mathbb{Z}_p^2)$
$[\![U]\!]$	$\{\, ((\mathbf{x},\mathbf{z}), (\mathbf{x}',\mathbf{z}')) \;:\; (\mathbf{x}',\mathbf{z}') = S_U(\mathbf{x},\mathbf{z}) \,\}$	$Q(\mathbb{Z}_p^2)^{\oplus n} \to Q(\mathbb{Z}_p^2)^{\oplus n}$

Clifford	**Affine symplectomorphism**
X	$S_X = \{\, ((x,z), (x+1, z)) : x, z \in \mathbb{Z}_p \,\}$
P	$S_P = \{\, ((x,z), (x, z+x)) : x, z \in \mathbb{Z}_p \,\}$
F	$S_F = \{\, ((x,z), (z, -x)) : x, z \in \mathbb{Z}_p \,\}$
C_X	$S_{C_X} = \{\, ((x_1, z_1, x_2, z_2), (x_1, z_1 - z_2, x_1 + x_2, z_2)) : x_i, z_i \in \mathbb{Z}_p \,\}$

Fig. 1. Sketch of denotational semantics of SPL, given without contexts.

$$\frac{\Gamma \vdash c \triangleright \Delta \quad \Delta \vdash d \triangleright \Sigma}{\Gamma \vdash c \,;\, d \triangleright \Sigma} \qquad \frac{}{\Gamma \vdash \mathbf{init}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma} \qquad \frac{}{\Gamma \vdash \mathbf{qinit}\ \underline{x} \triangleright \underline{x} : \mathtt{qpit}, \Gamma}$$

$$\frac{}{\Gamma \vdash \mathbf{skip} \triangleright \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma \vdash \underline{\mathbf{y}} {=} A {*} \underline{\mathbf{x}} \triangleright \underline{\mathbf{x}} : \mathtt{pit}^n, \underline{\mathbf{y}} : \mathtt{pit}^m, \Gamma}$$

$$\frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \mathbf{meas}\ \underline{x} \triangleright \underline{x} : \mathtt{pit}, \Gamma} \qquad \frac{}{\underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma \vdash \underline{\mathbf{x}} {*}{=} U \triangleright \underline{\mathbf{x}} : \mathtt{qpit}^n, \Gamma}$$

$$\frac{}{\underline{x} : \mathtt{qpit}, \Gamma \vdash \mathbf{disc}\ \underline{x} \triangleright \Gamma} \qquad \frac{}{\underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma \vdash \mathbf{ctrl}_P\ \underline{x}\ \underline{y} \triangleright \underline{x} : \mathtt{pit}, \underline{y} : \mathtt{qpit}, \Gamma}$$

Fig. 2. Formation rules for SPL. $n \in \mathbb{N}^{>0}$ and $\tau \in \mathrm{Ty}$, $\underline{\mathbf{x}} : \tau^n$ is shorthand for $\{\underline{x_1} : \tau, \cdots, \underline{x_n} : \tau\}$ such that $\underline{\mathbf{x}} = (\underline{x_1}, \cdots, \underline{x_n}) \in \mathrm{Reg}^n$. New variables are always assumed to be fresh.

We equip SPL with an *environment-transforming* type system, described in figure 2, which enforces linear usage of quantum data, and a denotational semantics in our category of hybrid classical/quantum symplectic relations, given informally in figure 1, proving that:

Theorem 3.1 (Full abstraction, informal). *Well-formed judgements c and d are observationally equivalent (in the quantum-mechanical sense) if and only if their relational semantics are equal.*

Nonlinear classical control

Given any function $f : \mathbb{Z}_p^n \to \mathbb{Z}_p^m$, we extend SPL with the base term $\underline{y} = f(\underline{x})$ and the formation rule

$$\frac{}{\{\underline{x} : \mathtt{pit}^n, \underline{y} : \mathtt{pit}^m, \Gamma\} \vdash \underline{y} = f(\underline{x}) \ \triangleright \ \{\underline{x} : \mathtt{pit}^n, \underline{y} : \mathtt{pit}^m, \Gamma\}}$$

$\Gamma \vdash \underline{y} = f(\underline{x}) \ \triangleright \ \Delta$ has a denotational semantics as a non-affine relation. In fact, it suffices to add a single relational to the denotational semantics of SPL which multiplies classical pits:

$$\left\{((a, b), a \cdot b) \in \mathbb{Z}_p^2 \oplus \mathbb{Z}_p\right\} : \mathbb{Z}_p^2 \to \mathbb{Z}_p \tag{2}$$

The non-affine classical control of stabiliser codes can produce mixed states which are no longer proportional to uniform mixtures of pure states; non-affine corrections between basis elements can create mixtures of pure states with different weights. Therefore, this extended denotational semantics cannot tell us the probability of measurement outcomes.

We weaken observational equivalence to forget about the probability distribution of measurements, recording only the *support* of measurements:

Definition 3.2. Say that two quantum channels are **nondeterministically observationally** equivalent in case they produce the same **possible** measurement outcomes according to the Born rule when acting on arbitrary density matrices.

This motivates the following conjecture:

CONJECTURE 3.3. *Well-formed judgements c and d in* SPL *are nondeterministically observationally equivalent if and only if* $[\![c]\!] = [\![d]\!]$.

4 Conclusion

We have developed a denotational semantics for stabiliser quantum programs based on the symplectic representation of stabiliser processes. This semantics captures the composition and manipulation of stabiliser codes, Pauli measurements, and classical affine control. By interpreting programs as relations in a hybrid classical and quantum symplectic setting, we obtain a compact model that is sound and fully abstract with respect to operational behaviour.

The semantics can be extended to include arbitrary classical control, representing the possible outcomes of Pauli measurements rather than their probabilities. Although this extension loses affine linearity, it still provides the necessary structure to reason about correctable and detectable errors in stabiliser quantum error correction.

In the qubit case, the affine symplectic representation no longer describes the full stabiliser theory. However, by restricting the available unitary operations to those generated by the controlled-not, Pauli, and swap gates, we recover the largest subcategory in which the symplectic representation remains valid. This fragment corresponds to CSS codes, which are widely used in fault-tolerant quantum computing.

The language presented in this paper is intentionally low-level. Future work will focus on developing higher-level programming abstractions that better reflect the underlying geometry of the semantics, such as constructs for graph states, graph-like operations, and error-correcting structures.

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