Towards a New Logic for Higher Order Quantum Computation

Julien Lamiroy
Université Paris-Saclay
CentraleSupélec
CNRS
ENS Paris-Saclay
Inria
Laboratoire des Méthodes formelles
Saclay, France

1 Introduction

The most standard way to represent a quantum process is the circuit model, where wires of qubits are modified by unitary boxes in a well defined order. With measurement and classical control, one can build many useful quantum operations, such as Shor's algorithm[10]. However, there exists a class of quantum processes that cannot be represented by straightforward circuits [3, 12]. In particular, the quantum switch [3], where the ordering of two input programs is determined by a qubit in a quantum way, has been realized physically.

While the more classical circuits are a well understood theory, a full, entirely rigorous model of these higher order processes, also known as superchannels, is still out of reach. Categorical constructions like the Caus[-] construction [11], which builds a category of causal processes from a compact-closed category, and profunctorial semantics [6], which builds a theory of "boxes with holes" built on a symmetric monoidal category, have been proposed to study these processes.

In addition to these high-level contributions, there are a number of languages that can encode many, if not all, of known higher-order processes that are outside the expressive power of standard circuits via quantum control [2, 1, 4]. However, these languages contain terms that do not preserve states, and therefore are outside of quantum theory. Moreover, there is usually no way to decide if a first-order diagram is a channel without computing its semantics, rather than a syntactic way.

Alongside this, the notion of control flow has been well studied in proof theory [2, 9, 11], often using Linear Logic's additive connectives. In fact, linear logic [5] presents multiple features that make it attractive to describe quantum operations, through its extremely restricted duplication and discarding of formulae that echo the no-cloning and no-discarding theorems of quantum theory, and the separation of contexts its multiplicative connectives enforce. In fact, many of the aforementioned quantum models and languages make use of structures close to linear logic, or to logics derived from it.

A natural question arises: can linear logic, specifically its multiplicative-additive fragment help us understand quantum superchannels and quantum control flow? This is not a new question, as it has al-

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ready been posed by the Many Worlds calculus [2]. However, as previously mentionned, not all of these processes are channel, with no known way to easily ensure this feature in a diagram.

In this work in progress, we attempt to give a new answer to this problem by developping a type system over both ground quantum data type and higher order types based on intuitionistic multiplicative additive linear logic. We introduce a new connective $\rightleftharpoons$ to encode unitary operations, the basis of quantum processes, and a novel form of sequent to render the reversible and dual properties of unitarity. We also exhibit a semantics for this new logic, along with its properties.

2 Contributions

We introduce a novel logic, Intuitionistic Unitary Linear Logic, based on linear logic to describe some higher order quantum processes, along with a semantics for its terms and its proofs.

2.1 Terms and Proofs

IULL is constructed on a two-tiered system of terms, following this grammar:

$$\begin{array}{rcl} A,B & := & 1 \mid a \mid A \otimes B \mid A \oplus B \\ F,G & := & A \rightleftharpoons B \mid F \otimes G \mid F \multimap G \mid F \,\&\, G \end{array}$$

The first line is ground term, representing quantum data. Their interpretation is straightforward. The second line is higher term, representing processes. A higher term F is represented by a vector space $\mathscr{V}_F$ and a set $[\![F]\!] \subseteq \mathscr{V}_F$. These relationships are described by this table:

H	$\mathscr{V}_H$	$[\![H]\!]$
$A \rightleftharpoons B$	$\mathrm{Lin}(A,B)$	unitaries
$F \otimes G$	$\mathscr{V}_F \otimes \mathscr{V}_G$	$\{f \otimes g; f \in [\![F]\!], g \in [\![G]\!]\}$
$F \multimap G$	$\mathrm{Lin}(\mathscr{V}_F, \mathscr{V}_G)$	$\{s; \forall f \in [\![F]\!], s(f) \in [\![G]\!]\}$
$F \,\&\, G$	$\mathscr{V}_F \oplus \mathscr{V}_G$	$\{(f,g); f \in [\![F]\!], g \in [\![G]\!]\}$

Examples

- $\underline{2} \rightleftharpoons \underline{2}$ represents unitaries over a single qbit.
- The quantum switch's type is represented as the term $((a \rightleftharpoons a) \otimes (a \rightleftharpoons a)) \multimap ((\underline{2} \otimes a) \rightleftharpoons (\underline{2} \otimes a))$
- The controlled execution $C : U,V \mapsto \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix}$ is represented by the term $((a \rightleftharpoons a) \,\&\, (a \rightleftharpoons a)) \multimap ((\underline{2} \otimes a) \rightleftharpoons (\underline{2} \otimes a))$. Here, the use of & allows to view this mapping as a linear map, which would not be possible with $\otimes$.
- $T_1 \multimap (T_1 \multimap T_2) \multimap T_3$ represents the (currified) application of a higher order map to some input of the correct type.

This intepretation of terms allows to construct an intepretation of proofs of IULL. The interpretation of a proof π is denoted as $[\![\pi]\!]$.

Theorem 1. *If F is a higher term, and π is a proof of F, $[\![\pi]\!] \in [\![F]\!]$.*

2.2 Theorems

Theorem 2 (Superunitarity). *If π is a proof of $(\bigotimes_i A_i \rightleftharpoons B_i) \multimap (C \rightleftharpoons D)$, then $[\![\pi]\!]$ is a superunitary, ie. a superchannel that conserves unitaries.*

While not total, this result aligns some of the processes extracted from proofs with physical theory.

The safety of the semantics being established, it is natural to examine what elements of terms are actually reachable by proofs.

Theorem 3 (Universality for Unitaries). *For all $n \geq 1$, for each unitary $\mathscr{U} : \mathbb{C}^n \to \mathbb{C}^n$, there exists π a proof of $\vdash \underline{n} \rightleftharpoons \underline{n}$ such that $[\![\pi]\!] = \mathscr{U}$.*

This is proven by proving the $n = 2$ case by showing that all rotations are realizable, and by then proving the controlled not is realizable. The rest follows by well known universality properties.

Most logics feature a cut rule, which allows to compose proofs together, in the same way processes can be composed sequentially. IULL also features cut rules acting in the same way.

Theorem 4 (Cut elimination for IULL). *Let π a proof of IULL. There exists ϖ a cut-free proof of IULL such that $[\![\pi]\!] = [\![\varpi]\!]$.*
Furthermore, we exhibit an explicit procedure to obtain such a ϖ.

The process of cut elimination links logics to theories of computation through the Curry Howard correspondance[7]. Eliminating instances of cut rules is analoguous to the reduction of terms into values, and is well studied.

3 Discussion

Due to the complex nature of IULL's terms, describing "channelness" beyond the terms laid out in Theorem 2 remains uncertain. However, categorical interpretation of superchannels may assist in a syntactical description of their properties, and extend Theorem 2.

Furthermore, while the quantum n-switch and the Grenoble process are realisable in our system, the Lugano process remains out of reach. We believe this failure to realise this process hints that the Lugano process presents a stronger form of indefinite causal ordering, and that further study of IULL would lead to a new classification of ICOs, maybe different from the hierarchy laid out in [8].

Finally, a logic with cut elimination may be a computational model, but it is unwieldy. In the same way well-known logics may be converted into lambda calculus and its variants, applying a similar process to IULL could lead to developping a more practical quantum lambda calculus that could be leveraged to describe higher order processes.

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