

Finite Observations, Infinite Behaviour: categorical semantics for stateful quantum processes

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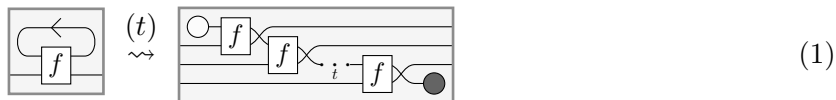
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Time-dependent processes are often described in terms of machines with an internal state which is updated as time evolves. However, from the perspective of an external observer, two such processes can produce the same behaviour, even when their internal states differ. We introduce a semantic construction which equates processes when they exhibit the same infinite external behaviour, as determined by all finite observations. Our construction is defined over any preorder-enriched monoidal category with a suitable notion of discarding. This provides a unified framework for partial, non-deterministic, probabilistic, and quantum processes evolving in space and time.

In this extended abstract we shall focus on the case of stateful quantum processes.

Stateful classical processes. State is a fundamental concept in computer science which is used to model processes which evolve over space and time. Examples of stateful processes include automata, stream transducers, and imperative programs with mutable variables.

Stateful processes are well-understood for *classical deterministic total* processes. For example, *Mealy machines* are a simple class of stateful processes. A Mealy machine consists of a transition function $f : S \times X \rightarrow S \times Y$ which at each time-step consumes an input in the set X , updates an internal state in the set S , and produces an output in the set Y [Mea55]. To run a Mealy machine for some finite amount of time, one must first initialise the state, repeatedly feed back the internal state to the transition function at the next time-step, and then discard the final state:



However, from this intensional description, it is not easy to tell when two Mealy machines will produce the same processes when they are run over time. This question is resolved using coalgebraic techniques (see [Rut00] for reference). A Mealy machine $f : S \times X \rightarrow Y \times S$ and an initial state $s_0 \in S$ determine a stream transducer $\text{run}(f, s_0) : X^{\mathbb{N}} \rightarrow Y^{\mathbb{N}}$ defined coinductively by the fixed point:

$$\text{run}(f, s_0)(x_0, x_1, \dots) = (y_0, \text{run}(f, s_1)(x_1, x_2, \dots)), \quad \text{where } (y_0, s_1) = f(s_0, x_0).$$

Two Mealy machines f and g , with initial states s_0 and t_0 , have the same *behaviour* over time when $\text{run}(f, s_0) = \text{run}(g, t_0)$.

Stateful quantum processes. Mealy machines, and more generally, stateful processes can be defined within arbitrary symmetric monoidal categories [DLdFR25], where different choices of symmetric monoidal categories capture different notions of computation. In this paper, we model quantum processes within the three following symmetric monoidal categories:

- The category CPM of completely positive maps between finite-dimensional matrix algebras;
- The subcategory CPTP of completely positive trace-preserving maps;
- The subcategory CPTNI of completely positive trace-nonincreasing maps.

CPTP captures the physical quantum operations. CPTNI captures quantum processes with post-selection, i.e., the physical quantum processes in addition to assertions. Finally, CPM captures the unnormalised quantum processes which in general are non-physical, but are nonetheless useful from a mathematical perspective.

Stateful processes built from these categories therefore capture different notions of quantum automata, quantum channels with memory, etc. (see for example [KW]).

However, just as for classical stateful processes, we can ask the same fundamental question:

When do two stateful quantum processes exhibit the same behaviour?

One may hope to use the fixed-point approach as before; however, the resulting category of streams is indiscrete for compact closed categories (Lemma 2.17) as well as categories with zero morphisms (Lemma 2.18). In other words, this approach doesn't capture the behaviour of stateful processes which may fail, or in which the future may be entangled with the past.

This is a big problem for modelling partial, nondeterministic, and quantum stateful processes. In particular, both CPM and CPTNI possess zero morphisms, where the former is moreover compact closed.

Approximation in quantum mechanics. To properly model the behaviour of stateful quantum processes we take inspiration from domain theory: we represent these processes not as fixed points in the above sense, but rather in terms of a series of increasingly better *approximations* [Sco70].

In order to define a reasonable notion of approximation in a symmetric monoidal category, we weaken Carboni and Walter's notion of a bicategory of relations [CW] to be able to model quantum processes where copying is not allowed:

Definition 0.1. A **discard bicategory** is a preorder-enriched symmetric monoidal category, such that for all objects X , there exists a **discard map** $\top_X : X \rightarrow I$, so that for all $f : X \rightarrow Y$,

$$\top_Y \circ f \subseteq \top_X, \quad \top_I = 1_I \quad \text{and} \quad \top_{X \otimes Y} = \top_X \otimes \top_Y.$$

We interpret $f \subseteq g$ to denote that a morphism g approximates f .

All of our aforementioned categories of quantum processes are discard bicategories with respect to the trace as the discard. However, there are two choices of preorder-enrichments:

Definition 0.2. Given two completely positive maps Φ and Ψ , recall that $\Psi \preceq \Phi$ in the **Löwner order** in case $\Psi - \Phi$ is completely positive.

It is well-known that this gives a poset-enrichment on our three categories:

Example 0.3. The Löwner order is a symmetric monoidal poset enrichment on CPM, CPTP and CPTNI.

It is immediate that CPTP is a discard bicategory with respect to the quantum trace and the Löwner order. Moreover, we show that this also induces a discard bicategory structure for quantum processes with postselection:

Proposition 0.4. *CPTNI is a discard bicategory with respect to the quantum trace and the Löwner order.*

However, this does not make CPM a discard bicategory. In order to endow CPM with this structure, we must use the following preorder introduced by Carette et al. [CdVP21, Definition 5].

Definition 0.5. Given two completely positive maps $\Phi, \Psi : \mathcal{H} \rightarrow \mathcal{K}$ in CPM, recall that $\Phi \sqsubseteq \Psi$ in **purification order** in case there exists some $\Psi_0 : \mathcal{H} \rightarrow \mathcal{S} \otimes \mathcal{K}$ and $\Phi_0 : \mathcal{S} \rightarrow \mathbb{C}$ such that

$$\Phi = (f_0 \otimes 1_{\mathcal{K}}) \circ \Psi_0, \quad \text{and} \quad \Psi = (\top_{\mathcal{S}} \otimes \mathcal{K}_Y) \circ \Psi_0.$$

The purification order takes this interpretation of the trace as being maximally informative seriously, expressing the fact that decohering a process reduces its information content. With respect to this order, it can easily be shown that:

Lemma 0.6. *CPM is a discard bicategory with respect to the purification order and the trace.*

Moreover, we show that for our two previous examples, both orders coincide:

Proposition 0.7 (Proposition 2.29). *The purification order and Löwner order coincide for CPTNI and CPTP.*

Therefore, in all three discard bicategories, we can interpret one process to approximate another in case it is noisier.

Categorical semantics for quantum stateful processes Using this structure of a discard bicategory, we formalise what is meant by a sequence of “increasingly better approximations”:

Definition 0.8. A sequence of morphisms in a discard bicategory,

$$\{f_t : X_0 \otimes \cdots \otimes X_t \rightarrow Y_0 \otimes \cdots \otimes Y_t\}_{t \in \mathbb{N}},$$

is **monotone** in case for all $t \in \mathbb{N}$,

$$(1_{Y_0 \otimes \cdots \otimes Y_t} \otimes \top_{Y_{t+1}}) \circ f_{t+1} \subseteq f_t \otimes \top_{X_{t+1}}.$$

In other words, we think of the element f_t as capturing incomplete knowledge about the stateful system at time t , such that progressing to the next time step $t + 1$ and forgetting the new output can only reveal additional constraints about the system at time t . This cannot reveal contradictory information, nor can it forget information from the previous time-step.

Carette et al. also use this notion of monotone sequence to model unnormalised stateful quantum processes [CdVP21]. However, because each element of the sequence is interpreted as a state of incomplete knowledge, two monotone sequences can exhibit the same behaviour over time while failing to be equal as sets. Moreover, the various equivalence relations which Carette et al. introduce fail to capture the equivalence of the behaviours of stateful quantum processes. For example, they are not able to identify processes when they fail.

We provide a mathematically elegant and simple solution to this problem: we assert that two sequences have the same behaviour in case they can be seen to approximate each other:

Definition 0.9. Given two monotone sequences $\mathbf{f}, \mathbf{g} : \mathbf{X} \rightarrow \mathbf{Y}$, say that \mathbf{g} *approximates* \mathbf{f} , written $\mathbf{f} \subseteq \mathbf{g}$, if for every $n \in \mathbb{N}$ there exists some **lookahead** $m \in \mathbb{N}$ such that:

$$(1_{Y_0 \otimes \dots \otimes Y_n} \otimes \top_{Y_{n+1} \otimes \dots \otimes Y_{n+m}}) \circ f_{n+m} \subseteq g_n \otimes \top_{X_{n+1} \otimes \dots \otimes X_{n+m}}.$$

Two monotone sequences are **observationally equivalent** in case they approximate each other.

In other words, two sequences are observationally equivalent if at some point in time, the information which is encapsulated by the one process will later be revealed by the other, and vice versa. This leads to another discard category whose morphisms are precisely the behaviours of stateful processes:

Definition 0.10. Given a discard bicategory \mathcal{C} , the category $\text{Obs}_{\mathbb{N}}(\mathcal{C})$ of **observational behaviours** has objects given by infinite sequences of objects in \mathcal{C} and morphisms given by observational equivalence classes of monotone sequences.

Indeed, this construction preserves the structure needed to define approximation:

Theorem 0.11 (Theorem 3.7). *$\text{Obs}_{\mathbb{N}}(\mathcal{C})$ is a discard bicategory.*

We show that for all of our examples, the category of behaviours handles failures properly, and behaves well in the presence of compact closure. This is evidence for the canonicity of our construction, as no other comparable semantics can accommodate this structure.

Modelling more general notions of state. The state of a computational process need not evolve linearly over time. For example, cellular automata are machines whose state is described both by state and time: Conway's game of life has states parametrised by the set $\mathbb{N} \times \mathbb{Z}^2$.

We generalise the notion of state using the following definition:

Definition 0.12. $\mathcal{J} \subseteq \mathcal{P}_f(I)$ is **upward directed**, in case for any pair of contexts $j, j' \in \mathcal{J}$ there exists some $j'' \supseteq j \cup j'$ in \mathcal{J} .

The set $\mathcal{P}_f(I)$ is interpreted as the shape of the ambient discrete space; whereas the set \mathcal{J} indexes the finite contexts with which this space can be observed. We can define an analogous notion of monotone sequences:

Definition 0.13. Let \mathcal{C} be a preorder enriched discard category and let $\mathbf{X}, \mathbf{Y} \in (\text{ob}(\mathcal{C}))^I$ be a set of objects in \mathcal{C} indexed by I . A **monotone net** $f : \mathbf{X} \rightarrow \mathbf{Y}$ in \mathcal{C} is an assignment to every $j \in \mathcal{J}$, a morphism $f_j : \bigotimes_{i \in j} X_i \rightarrow \bigotimes_{i \in j} Y_i$, such that for all $j \subseteq j' \in \mathcal{J}$:

$$(1_{Y_j} \otimes \top_{Y_{j' \setminus j}}) \circ f_{j'} \subseteq f_j \otimes \top_{X_{j' \setminus j}}, \quad \text{where } Y_j := \bigotimes_{k \in j} Y_k \text{ for any } j \in \mathcal{J}.$$

By taking $I := \mathbb{N}$ and \mathcal{J} to be the set $\{[0, t] \mid t \in \mathbb{N}\}$ of finite closed intervals including 0, a monotone net is exactly a monotone sequence. The natural numbers index time, whereas the intervals $[0, t]$ express the regions of time which can be observed, where one always starts from the beginning of time and is not allowed to forget anything.

The previous machinery for monotone sequences extends straightforwardly in the obvious way, leading to the following construction:

Definition 0.14. Given a discard bicategory \mathcal{C} and an upward directed set $\mathcal{J} \subseteq \mathcal{P}_f(I)$, the discard bicategory $\text{Obs}_{I, \mathcal{J}}(\mathcal{C})$ of observational behaviours has objects given by infinite sequences of objects in \mathcal{C} and morphisms given by observational equivalence classes of monotone nets.

Future directions. The techniques which we use to give a categorical semantics for stateful quantum processes bear a very close resemblance to domain theory. For example, consider the category $\text{Rel}(\text{KHaus})$ of closed relations between compact Hausdorff spaces. We prove a non-deterministic compactness theorem: showing that given cofinal $\mathcal{J} \subseteq I$ there is a functor of discard bicategories $\text{Obs}_{I,\mathcal{J}}(\text{Rel}(\text{KHaus})) \rightarrow \text{Rel}(\text{KHaus})$ (Theorem 5.5). We prove this fact using standard topological techniques; however, we suspect that there is perhaps some more general, deeper structure at play.

The lattice of closed subspaces of a topological space is a directed complete partial order (DCPO); therefore $\text{Rel}(\text{KHaus})$ is DCPO-enriched. DCPO-enrichment is the basic structure needed to interpret recursive definitions in domain theory [SA94]. Moreover, cofinal nets appear throughout the domain theory literature. We suspect that DCPO enrichment and cofinality are the precise ingredients needed to prove such compactness theorems in general:

Conjecture 0.15. *Given a DCPO-enriched discard bicategory \mathcal{C} and a cofinal set $\mathcal{J} \subseteq \mathcal{P}_f(I)$, there is a functor of discard bicategories: $\text{Obs}_{I,\mathcal{J}}(\mathcal{C}) \rightarrow \mathcal{C}$.*

This leads to the natural question: is there some quantum analogue to this nondeterministic compactness theorem? In this paper, we have given an algebraic description of “infinite” stateful quantum processes in the Schrödinger picture. However, these processes are not truly infinite *dimensional* in the standard sense, as they lack the appropriate analytic structure. Neither three of our examples extend to discard bicategories in infinite dimensions because the trace fails to be bounded operator. However, in the Heisenberg picture the category $\mathbf{W}^*\mathbf{Alg}_{\text{NCPSU}}$ of W^* -algebras and normal completely positive subunital maps is DCPO-enriched [Ren14], where the *unit* is a bounded operator. Therefore, by working in the dual picture:

Conjecture 0.16. *$\mathbf{W}^*\mathbf{Alg}_{\text{NCPSU}}^{\text{op}}$ is a discard bicategory. Moreover, given a cofinal set $\mathcal{J} \subseteq \mathcal{P}_f(I)$, there is a functor of discard bicategories: $\text{Obs}_{I,\mathcal{J}}(\mathbf{W}^*\mathbf{Alg}_{\text{NCPSU}}^{\text{op}}) \rightarrow \mathbf{W}^*\mathbf{Alg}_{\text{NCPSU}}^{\text{op}}$.*

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