Noncommutative models of quantum computing

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Abstract

We present a different vision on the foundations of quantum computing, which we believe should be based on the principles of noncommutative geometry. This requires a higher level of abstraction than is common in quantum computing. In particular, working with infinite dimensional systems is necessary. We argue why this level of abstraction is necessary to make the next step in quantum computing, and give an overview of applications of noncommutative structures in quantum information science and in the denotational semantics of quantum programming languages and logics in particular.

I. The need for and use of abstraction in computer science

In essence, since classical computing concerns the manipulation of finite, discrete data in the form of bits, it can be described by finitary mathematics. Nevertheless, more abstract models of computing using higher mathematics such as set theory or category theory are common for several reasons:

- Higher mathematics allows us to **prove** statements about computation. For instance, even though the Halting Problem is undecidable, proving that specific classes of programs will always terminate typically relies on techniques from higher mathematics. This mirrors Gödel's First Incompleteness Theorem, which asserts that some statements about finitary mathematics cannot be proven within a finitary framework, and requires higher mathematics.
- More abstract models give insights in how programs can safely be **composed**, turning a "wall of code" into a "library of parts". The modular character of these models allow us to uncover computational patterns, and hence lead to new insights in the design of programs and algorithms.

A. The role of denotational semantics

The "library of parts" mentioned above is not merely a conceptual metaphor; it is grounded in the formal framework of *denotational semantics*, in which programming constructs are translated to mathematical objects that can be composed. Moreover, denotational semantics not only offers ways to construct models with a modular character; the method itself has a modular character. It explains which computational features correspond to what mathematical structures and how these structures can be combined in order to produce a model for a language that combines these features. As a consequence, as Abramsky notes in [1], denotational semantics provides a computational paradigm that exists independently of any specific language or calculus. Studying the mathematical structures relevant to denotational semantics yields new insights into and even new paradigms of programming language design.

This impact of denotational semantics on the design of programming languages is extensive and well-documented. For instance, Moggi's discovery of the description of computational effects via monads [2], [3] provides a mathematical foundation for the question which effects can be combined, and how effects can be transformed. Another example is provided by linear logic, which Girard distilled from the coherent space model [4], [5]. This is in particular relevant for quantum computing, since type systems of quantum programming languages are typically linear.

B. Abstracting away from quantum circuits

Current quantum computation is expressed at the circuit level, analogous to assembly code. To reason compositionally and build higher-level abstractions, we need models where computations can pass programs as data, branch and loop coherently on quantum information, and expose causal structure. This requires moving beyond classical control flow toward strictly quantum control structures. Specifically, this involves the development of a `quantum if` and `quantum case` for branching, as well as a `quantum while` loop for iterative processes that can terminate in superposition. In particular, the implementation of `quantum while` is challenging, as it involves loops that can exist in a superposition of "running" and "terminated". This forces a departure from the classical intuition of a discrete halting state, as any attempt to observe termination in a standard way risks collapsing the very coherence the program seeks to exploit. Consequently, it remains a profound challenge to define a termination condition that is both mathematically rigorous within a denotational framework and physically realizable without destroying the quantum advantages of the computation.

In particular, it is necessary to understand the quantum correlations between higher-order quantum operations, which is also essential in tasks such as channel discrimination [6], quantum metrology [7], programmable processors [8], and processes with indefinite causal order [9], [10]. The latter are processes corresponding to quantum programs where the order of application of subroutines exists in coherent superposition. The prime example of such a process is the the quantum switch [11], which is a higher-order process with as inputs two operations A and B that are applied to a qubit, but the order of application is controlled by a second "control" qubit. Effectively, the quantum switch is an example of a process that could be programmed via a `quantum case` statement, where we apply $A \circ B$ if the state of the control qubit is $|0\langle$ and $B \circ A$ if the state of the control qubit is $|1\rangle$. If the state of the control qubit is in a superposition of $|0\rangle$ and $|1\rangle$, so is the order of application of A and B.

C. The issue with current denotational models of quantum computing

Most denotational models of quantum programming languages are constructed in the following way. The cornerstone is a category $\mathbf{M}$ of finite-dimensional objects such as Hilbert spaces or operator algebras. The morphisms form a class of first-order quantum processes such as quantum circuits or quantum channels. Because $\mathbf{M}$ alone cannot support complex programming features, it is typically embedded into a larger category $\mathbf{C}$ that is obtained via categorical constructions on $\mathbf{M}$ such as presheaves. However, this approach introduces two fundamental issues:

- **Unphysical morphisms:** The category $\mathbf{C}$ is often "too large", containing a vast number of morphisms that do not correspond to to any physical operation. This forces the model to treat higher-order code as a mere calculation that must eventually "collapse" back into a simple circuit in $\mathbf{M}$ to be meaningful.
- **The quantum correlation gap:** Because constructions like presheaves are built on the logic of sets, they inherit a classical framework that is fundamentally at odds with the non-local nature of quantum mechanics. As a consequence, even when these models do capture physical higher-order operations, they fail to account for the quantum correlations between them, which are necessary to describe operations such as the quantum switch.

D. The noncommutative alternative

To move beyond these limitation, we need to construct models completely in terms of mathematical framework suitable for the description of quantum physics. In this way, we ensure that every level of the program remains "natively quantum", preserving the essential correlations that classical categorical constructions inevitably fail to encode.

Noncommutative geometry (NCG) [12] provides such a framework by replacing the algebra of functions on a space (which determines the space) with an *operator algebra*, i.e., an algebra of operators on a Hilbert space. More generally, NCG concerns the generalization of mathematical structures to the noncommutative setting of operators on Hilbert spaces, a process also called *quantization*. In this dictionary:

- *C*-algebras* are quantized locally compact Hausdorff spaces;
- *von Neumann algebras* are quantized measure spaces.

The **main thesis** of this contribution is that models for quantum computing should completely be described in the language of NCG.

This comes at a cost of working more abstractly. However, similar arguments as at the beginning of the section apply: we expect that working on a higher abstraction level is necessary to prove stronger statements about quantum computing and to obtain the insights into potential new computational paradigms. NCG offers those insights via the following principles:

- **Quantization:** Many quantum phenomena have classical counterparts. Quantizing the mathematical structures describing those counterparts often yields suitable structures to model the quantum phenomena.
- **Pure quantum features**, i.e., features without a classical counterpart, typically naturally emerge from the noncommutative structures underlying these models.

The most spectacular feature that emerges naturally from the noncommutative structure of a model is the Higgs particle, which has to be added manually to the Standard Model of particle physics, whereas it naturally appears in the Noncommutative Standard Model [13]. It is to be expected that if pure quantum constructs as `quantum if`, `quantum case` and `quantum while` can be implemented, the necessary structure will be noncommutative.

Finally, NCG offers insights in why a structure is not suitable to model a certain features. For example, the first example of a completely noncommutative model for a quantum programming language was the von Neumann algebras model of Cho and Westerbaan [14]. This model is dcpo-enriched [15], [16], which suggests it supports recursion, but all attempts to establish a formal proof have thus far proven unsuccessful. NCG explains why, because in its dictionary, von Neumann algebras correspond to measure spaces, which do not come along with a partial order suitable for recursion. Hence it is not to be expected that noncommutative measure spaces, i.e., von Neumann algebras, support recursion. We will see in the next section how NCG also offers guidance in finding the right structure to model recursion.

II. Discrete quantization

One way to find noncommutative generalizations of mathematical structures is via a process called *discrete quantization*. Central in this approach is the category $\mathbf{qRel}$ whose objects are (possibly infinite) sums of matrix algebras. These objects generalize sets, and are therefore called *quantum sets*. The morphisms of $\mathbf{qRel}$ generalize binary relations and are therefore called *quantum relations*. $\mathbf{qRel}$ has a categorical structure resembling that of the category $\mathbf{Rel}$ of sets and binary relations:

- Both categories are *dagger compact*, the type of categories central in the program of Categorical Quantum Mechanics [17], [18], [19];
- Both categories are *quantaloids*, i.e., homsets are complete lattices and composition preserves suprema. For $\mathbf{Rel}$, the quantaloid structure is obtained by ordering parallel binary relations by inclusion.

Most mathematical structures can be described in terms of sets and binary relations satisfying some constraints in the language of dagger compact quantaloids. Hence, often a structure can be generalized to the noncommutative setting by replacing any instance of a set in its definition by a quantum set and any binary relation by a quantum relation, while requiring that the same constraints hold. As a simple example, a function $f : X \to Y$ between sets is a binary relation satisfying $f^\dagger \circ f \geq \mathrm{id}_X$ and $f \circ f^\dagger \leq \mathrm{id}_Y$, and hence we can define a *quantum function* $F : \mathcal{X} \to \mathcal{Y}$ between quantum sets to be a quantum relation satisfying $F^\dagger \circ F \geq \mathrm{id}_{\mathcal{X}}$ and $F \circ F^\dagger \leq \mathrm{id}_{\mathcal{Y}}$. Discrete quantization yields various noncommutative structures that are relevant to quantum information science, and some already existing notions can also be understood via the methodology. We give some examples:

- **Quantum graphs** were introduced for quantum error correction [20], and can be understood as discrete quantizations of ordinary graphs [21]. Ordinary graphs can be regarded as quantum graphs whose underlying quantum set is commutative. Also homomorphisms between graphs can be generalized to the noncommutative setting, resulting in the notion of a quantum homomorphism, which is used to study nonlocality via homomorphism games [22], [23]. In particular, it was shown that there are ordinary graphs G and H for which there is no homomorphism $G \to H$, but there exists a quantum homomorphism $G \to H$, corresponding to the existence of a winning quantum strategy.
- **Quantum metrics** were introduced by Kuperberg and Weaver [24], and include as an example the quantum Hamming metric used in quantum error correction. The discrete quantization method stems from this work, as Weaver distilled the notion of a quantum relation from this work [25].

Generally, discrete quantization seems to be subject to the following pattern: given a category $\mathbf{C}$ of some mathematical structure, quantizing this structure yields a category $\mathbf{qC}$ that has similar properties as $\mathbf{C}$, except if $\mathbf{C}$ is cartesian monoidal, then $\mathbf{qC}$ is noncartesian monoidal, reflecting the quantum character of its objects.

A. Denotational models via discrete quantization

The category $\mathbf{Set}$ provides a standard model for pure computation without recursion, using its cartesian closed structure to support higher-order processes and monads to handle effects. For example, subprobabilistic processes can be modelled as Kleisli maps of the countable subdistributions monad D on $\mathbf{Set}$.

Quantum sets follow a parallel trajectory. The category $\mathbf{qSet}$ of quantum sets and quantum functions is symmetric monoidal closed, supporting quantum function types [26]. Just as classical functions lift to stochastic processes via D, quantum functions $F : \mathcal{X} \to \mathcal{Y}$, which correspond to unital $*$-homomorphisms $\mathcal{Y} \to \mathcal{X}$, lift to quantum channels via a quantum subdistribution monad $\mathcal{D}$ on $\mathbf{qSet}$. The Kleisli maps of $\mathcal{D}$ correspond to completely positive subunital maps, and the CP-Löwner order of quantum information theory emerges at the counterpart of the pointwise order of subdistribution, providing a natural framework for impure quantum computation.

B. Quantum cpos as a step towards a quantum domain theory

In collaboration with Kornell and Mislove, I introduced *quantum cpos*, which are quantum sets equipped with a ω-complete quantum order, i.e., a quantum endorelation that is the noncommutative generalization of an ω-complete partial order. Traditionally, ω-complete partially ordered sets (cpos) are used to model ordinary recursive pure computations, and similarly we showed that recursive pure quantum computations can be described in a model of quantum cpos [27].

This suggests the existence of a quantum domain theory based on quantum cpos that offers systematic methods to construct models for programming languages with recursion, mirroring the classical situation where cpos form the foundation of domain theory.

However, the step to modelling impure quantum computation is challenging. Classically, this is done via probabilistic powerdomain monads on the category $\mathbf{CPO}$ of cpos and Scott continuous maps. In the quantum setting, there are two stages of computation: circuit generation and circuit execution. The Löwner order describes the progress of a computation during circuit execution, whereas the ω-complete quantum orders of quantum cpos describe the progress during circuit generation. A quantum probabilistic powerdomain monad should combine both orders in a compatible way, which is still an open problem.

III. Operator spaces

Another noncommutative approach to the denotational semantics of is based *Banach spaces*, i.e., complete normed vector spaces (which we assume to be complex valued). These spaces form the foundational structures of quantum physics. For example, a quantum system is typically described by a Hilbert space H, which is an instance of a Banach space. Two other examples of Banach spaces are:

- $B(H)$, the algebra of all bounded operators on H. Since this space is spanned by the selfadjoint operators, representing the system's observables, it is the natural setting for the *Heisenberg picture* of quantum physics, where the system's time evolution is described in terms of observables.
- $T(H)$, the space of trace-class operators on H. Since it is spanned by the density operators, representing the (mixed) states of the system, this space is fundamental to the *Schrödinger picture* of quantum physics, in which time evolution is described in terms of states.

When equipped with the Banach space projective tensor product $\overset{p}{\otimes}$, the category $\mathbf{Ban}$ of Banach spaces and contractions becomes a model for intuitionistic linear logic. Firstly, $(\mathbf{Ban}, \overset{p}{\otimes}, \mathbb{C})$ is symmetric monoidal closed, where the internal hom of Banach spaces X and Y is given by the space $B(X,Y)$ of all continuous linear operators $X \to Y$. Furthermore, it has *Lafont exponentials*. Writing $\mathbf{CoComon}$ for the category of cocommutative comonoids internal to $(\mathbf{Ban}, \overset{p}{\otimes}, \mathbb{C})$, this means that the forgetful functor $\mathbf{CoComon} \to \mathbf{Ban}$ has a right adjoint. The induced comonad models the exponential modality of linear logic. The existence of the adjunction follows because $\mathbf{Ban}$ is *locally presentable* and symmetric monoidal closed using results from [28]. Recall that a category is locally presentable if it is locally small, cocomplete, and there is a cardinal λ and a small set of λ-compact objects that generate the category via λ-filtered colimits [29], [30].

However, $\overset{p}{\otimes}$ does not correctly describe the composition of quantum systems, since the Hilbert space tensor product $H \otimes K$ is not equal to $H \overset{p}{\otimes} K$ (in general in infinite-dimensions). Furthermore, in general, we have $T(H \otimes K) \not\cong T(H) \overset{p}{\otimes} T(K)$ and $B(H \otimes K) \not\cong B(H) \overset{p}{\otimes} B(K)$. The reason for this mismatch with quantum theory is that Banach spaces are essentially classical mathematical structures.

A. Operator spaces

The solution is to look at noncommutative versions of Banach spaces: *operator spaces*. These form a well established notion in NCG and are defined as follows:

Definition III.1. [31], [32] A (concrete) *operator space* is a norm-closed subspace $X \subseteq B(H)$ for some Hilbert space H. If $Y \subseteq B(K)$ is another operator space, a linear map $u : X \to Y$ is called *completely bounded* if there is some Hilbert space $\hat{H}$, a unital $*$-homomorphism $\pi : B(H) \to B(\hat{H})$ and bounded linear maps $v_1, v_2 : K \to \hat{H}$ such that $u(x) = v_2^* \pi(x) v_1$ for each $x \in X$. If v_1 and v_2 can be chosen to be contractions, then u is called a *complete contraction*. If $v_1 = v_2$, then u is called *completely positive*. We write $\mathbf{OS}$ for the category of (possibly infinite-dimensional) operator spaces with complete contractions as morphisms.

Any Banach space X is isometrically embeddable into $B(H)$ for some Hilbert space H, and can hence be regarded as an operator space. However, one has to keep track of the embedding as various choices are possible. Any such choice is called an *operator space structure (OSS)* on X. Equivalently, one can define operator spaces as Banach spaces equipped with an OSS. Both $T(H)$ and $B(H)$ have a canonical OSS, and can be regarded as operator spaces. The *operator space projective tensor product* $\hat{\otimes}$ on $\mathbf{OS}$ is the operator space counterpart of $\overset{p}{\otimes}$ on $\mathbf{Ban}$. We prove that $\mathbf{OS}$ is symmetric monoidal closed with respect to $\hat{\otimes}$ with internal hom $CB(X,Y)$, the operator space of completely bounded maps $X \to Y$. Moreover, $\hat{\otimes}$ correctly describes the composition of quantum systems in the following sense: any Hilbert space H becomes an operator space H_c when equipped with a canonical OSS that captures the inner product of H and then $(H \otimes K)_c \cong H_c \hat{\otimes} K_c$, and $T(H \otimes K) \cong T(H) \hat{\otimes} T(K)$.

It is straightforward to see that $(\mathbf{OS}, \hat{\otimes}, \mathbb{C})$ is symmetric monoidal closed. The internal hom of operator spaces X and Y is given by the space $CB(X,Y)$ of all completely bounded linear operators $X \to Y$. Zamdzhiev and I proved the following:

Theorem III.2. *The category* $\mathbf{OS}$ *is locally presentable and therefore a model of Intuitionistic Linear Logic.*

B. BV-logic and the Haagerup tensor product

The operator space projective tensor product $\hat{\otimes}$ is only one of the many tensor products on $\mathbf{OS}$. The *Haagerup tensor product* $\overset{h}{\otimes}$ is perhaps the most interesting amongst those tensor products, because is a nonsymmetric monoidal product on $\mathbf{OS}$, which furthermore has no Banach space counterpart. Hence, it is an example of pure quantum effect that appears from the noncommutative structure of operator spaces. The Haagerup product satisfies the following property: for any two operator spaces X and Y there is a complete contraction $\iota : X \hat{\otimes} Y \to X \overset{h}{\otimes} Y$ such that for each complete contraction $\varphi : X \hat{\otimes} Y \to B(H)$, we have that φ factorizes via ι if and only if it can be written as $\varphi(x \otimes y) = \psi_1(x)\psi_2(y)$ for complete contractions $\psi_1 : X_1 \to B(K,H)$ and $\psi_2 : X_2 \to B(H,K)$, i.e. if it exhibits a sequential order. In [33], Zamdzhiev and I showed that the pure state quantum switch does not factorize via ι, confirming that it cannot be decomposed in a reasonable sequential way.

The Haagerup tensor product provides a way to identify the sequential character of a quantum operation. This capability could prove essential for implementing a guarded recursion operator within a quantum control flow. The Haagerup does not model a connective of linear logic, but it seems to have many similarities with the seq connective of *BV-logic*. This logic is an extension of Multiplicative Linear Logic (MLL) that describes sequentiality of operations. In fact, in [34], Li and Zamdzhiev showed that the category $\mathbf{FdOS}$ of finite-dimensional operator spaces is a model of the exponential-free fragment of BV-logic, where $\overset{h}{\otimes}$ indeed models $\mathtt{seq}$. An intuitionistic version of BV-logic was recently introduced [35]. It is doubtful whether $\mathbf{OS}$ and $\overset{h}{\otimes}$ model intuitionistic BV-logic, we aim to investigate whether refinements such as dual operator spaces (operator spaces that are dual to another operator space) and the normal Haagerup tensor product $\overset{\sigma h}{\otimes}$ are more promising. Since dual operator spaces are also expected to be dcpo enriched, they might form an interesting setting for studying recursion with quantum control.

We note that this submission is partially based on [33] and [34], which are also both submitted for talks to this workshop. This submission is more intended as an overview talk. The other submissions (including companion papers and appendices) span more than 100 pages, so there is more than enough material to present during the workshop. Should all submissions be accepted for talks, the presenters will synchronise with each other in order to minimise overlap.

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