Completeness Is Not Enough: Simpler Presentations and Minimality for Near-Clifford Circuit Fragments

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***Abstract*—Completeness for a quantum-circuit fragment answers only part of the equational story. It tells us that every valid identity can be derived, but it does not distinguish genuine algebraic laws from equations that merely encode structural symmetry, nor does it say whether the surviving axioms are independent. We therefore revisit six near-Clifford fragments in a PROP setting, where wire permutations are ambient structure rather than fragment-specific axioms, and obtain smaller complete presentations for qubit Clifford, real Clifford, CNOT-dihedral, bounded-wire Clifford+T, bounded-wire Clifford+CS, and qutrit Clifford.**

Across the six fragments, the PROP reformulation exposes a smaller algebraic core while still leaving nontrivial derivations to recover laws that were previously explicit. The reduced PROP presentations for qubit Clifford, real Clifford, and CNOT-dihedral are minimal; those for Clifford+T and Clifford+CS are minimal at low arity but leave one further wire layer to settle; and qutrit Clifford is minimal up to two qutrit wires, leaving only a single ternary braid relation unresolved.

***Index Terms*—quantum circuits, equational theories, PROPs, axiom independence, minimality**

I. Introduction

Quantum-circuit optimisation and verification rely heavily on equational reasoning: a subcircuit may be replaced by another whenever a finite rule set proves that the two diagrams denote the same unitary. For several near-Clifford fragments, including qubit Clifford, real Clifford, CNOT-dihedral, bounded-wire Clifford+T, bounded-wire Clifford+CS, and qutrit Clifford, complete finite presentations are already known [18], [16], [1], [3], [4], [14]. These fragments sit between two broader lines of work: richer circuit languages now have complete syntactic calculi, while minimal circuit theories and near-minimal ZX axiomatisations show that, once completeness is available, independence becomes the next structural question [9], [8], [7], [19], [2]. Because unrestricted circuit languages may require indispensable rules of unbounded arity [7], near-Clifford fragments remain a useful testbed: their presentations stay finite, but minimality is already nontrivial.

Completeness, however, settles only part of the story. A presentation may still mix two rather different kinds of equations: laws that express genuine algebraic interaction inside the fragment, and laws whose main role is to encode structural symmetry or other presentational overhead inherited from a particular syntax. That distinction matters in practice: automated rewriting and verified optimisation benefit from smaller rule sets, and unnecessary axioms enlarge search spaces without adding expressive power [12], [10]. More importantly, once the structural layer has been peeled away, one can ask which remaining laws are actually indispensable.

We study six near-Clifford fragments through a common PROP reformulation. In this setting symmetry is ambient structure, so swaps and permutations belong to the graphical language instead of being encoded by fragment-specific equations [15], [11], [17], [13]. The present extended abstract distils the six-fragment analysis of [6]. The point is not that PROP syntax alone explains the smaller rule counts: some omitted equations are structural, while others are genuine fragment identities that must be recovered by derivation from a smaller core. The main questions are therefore whether those reduced cores remain complete and which surviving axioms are independent.

Table I summarizes the resulting reductions and current minimality frontiers. Section II develops the shared reduction pattern through the qutrit running example, and Section III turns to minimality via separating models.

II. PROP Reduction Across the Fragments

Many published circuit calculi start from a PRO-style syntax. Sequential and parallel composition are primitive, while wire permutations appear as explicit circuit equalities. We work instead in a PROP, where symmetry lives in the ambient graphical language. This gives the six fragments a common interface and cleanly separates fragment-specific algebra from structural symmetry. The point is not that PROP syntax alone explains the smaller rule counts. Rather, it changes what has to be shown: once structural rules are no longer counted as fragment axioms, one can ask which earlier fragment identities remain primitive and which can be recovered from a smaller core. This perspective is comparative from the outset. For qubit Clifford, real Clifford, and CNOT-dihedral, the reduction stops at one- and two-wire structure; for bounded-wire Clifford+T and Clifford+CS, only a small higher-wire layer remains; qutrit Clifford exhibits the same stratification but keeps one ternary braid relation. We therefore display the qutrit core not as an exceptional case, but as one fragment in which the common pattern fits in a single figure.

Figure 1 will serve as the running example in the sequel. It displays the reduced qutrit PROP calculus itself: local laws for ω, H, and S, four binary CNOT interaction laws including the swap decomposition, and the lone ternary braid relation I.

TABLE I
REDUCED COUNTS AND MINIMALITY STATUSES ARE ESTABLISHED IN [6]; IN THE PROP SETTING, STRUCTURAL SYMMETRIES ARE AMBIENT AND ARE THEREFORE NOT COUNTED AS FRAGMENT AXIOMS.

fragment	earlier complete presentation	reduced PROP core	minimality status
qubit Clifford	15 rules in [18]	8 rules in [6]	minimal
real Clifford	16 rules in [16]	10 rules in [6]	minimal
qutrit Clifford	18 rules in [14]	10 rules in [6]	minimal up to 2 qutrit wires
Clifford+T on up to 2 qubits	18 rules in [3]	11 rules in [6]	minimal up to 1 qubit
Clifford+CS on up to 3 qubits	17 rules in [4]	14 rules in [6]	minimal up to 2 qubits
CNOT-dihedral	13 rules in [1]	11 rules in [6]	minimal

Reduced qutrit PROP calculus [6]

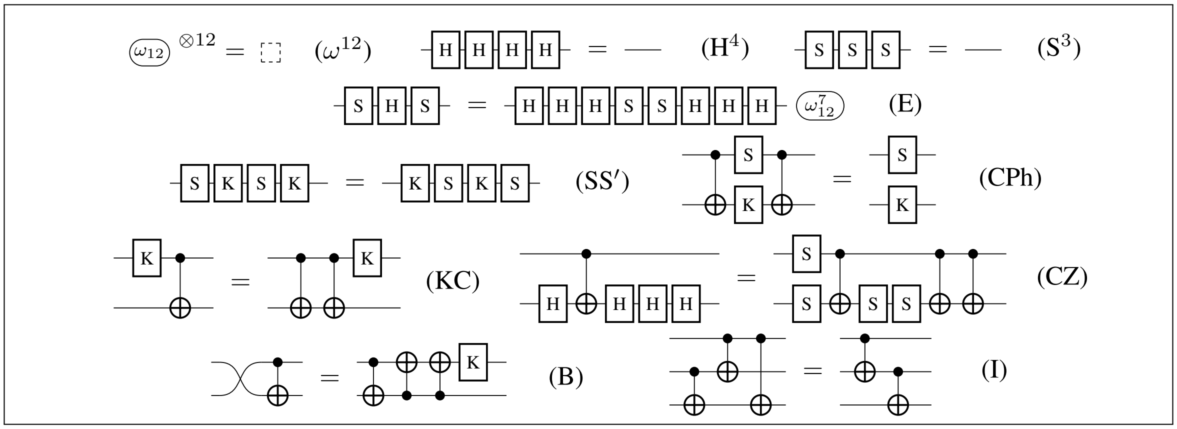

Fig. 1. The qutrit fragment is generated by the phase ω, the one-qutrit gates H and S, and qutrit CNOT, with $K = H^2$ used as shorthand; the axioms split into local laws, a short binary interaction layer, and one ternary braid relation; in the PROP setting, permutation rules are structural and further equations from the older PRO presentation become derivable from this core.

III. MINIMALITY VIA SEPARATING MODELS

Once completeness has been transferred into the common PROP setting, the next question is whether every surviving rule is really needed. A complete presentation may still hide redundancy, with some equations already forced by the others and therefore adding nothing essential to the logic of the fragment. We formalise this issue through minimality: an axiom is independent when it cannot be recovered from the rest, and a complete finite presentation is minimal when every axiom has that property.

The proof method is semantic. To test a candidate relation ρ, one removes ρ, interprets the remaining generators into a small target PROP, and checks that all retained equations continue to hold while the two sides of ρ become distinct. By Birkhoff's separation principle [5], such a model witnesses independence. In these circuit fragments the method is helped by a simple structural fact: every generator is an endomorphism, so derivations preserve wire count. As a result, a unary or binary rule can be separated inside a truncated one- or two-wire system, without having to account for arbitrarily many additional wires.

This viewpoint also clarifies what the reduced calculi achieve. When such separators exist for every axiom, the surviving rules are exactly the semantically indispensable ones at the arities under consideration. The bounded-wire fragments tell a slightly different story. Completeness is known up to two qubits for Clifford+T and three qubits for Clifford+CS, whereas the current separators certify minimality only up to one and two qubits respectively. The low-arity cores are already irredundant, but the unresolved cases call for richer semantic targets.

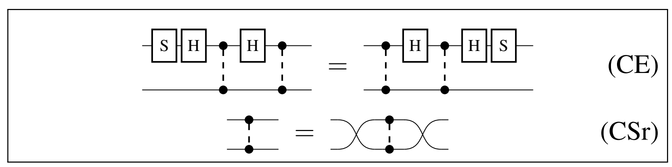

Fig. 2. Two two-qubit Clifford+CS axioms that may look more like rewriting conveniences than indispensable algebra: the rule CE and the symmetry rule CSr, which amounts to $\mathrm{SWAP} \circ CS = CS \circ \mathrm{SWAP}$.

These two examples show that the reduced two-qubit Clifford+CS core still contains non-obvious independent interaction laws [6].

At three-qubit arity, the remaining Clifford+CS source rules collapse to this single indexed family, so the current gap looks more like a missing separator than like obvious surplus in the reduced calculus.

Across the six fragments, the minimality outcomes fall into three profiles. For qubit Clifford, real Clifford, and CNOT-dihedral, small recurring occurrence and counting models

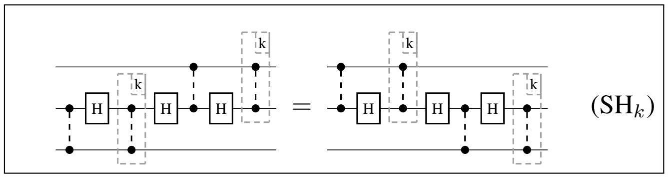

Fig. 3. The unified three-qubit Clifford+CS rule from [6]. The dashed box is a repetition box indexed by $k \in \{0,1,2,3\}$, so one schematic equation collects the remaining three-qubit source family.

already separate every surviving axiom, so the reduced PROP cores are fully minimal. For bounded-wire Clifford+T and Clifford+CS, the same low-arity methods settle the unary and binary layers, but completeness extends one wire further than the current separators. Qutrit Clifford sits between these situations: every unary and binary axiom is already separated, so only one ternary braid relation remains open.

One recurring separator family is given by counting maps. For a chosen generator g and modulus k, the interpretation $\#\{g\}_{[k]}$ records how many copies of g occur in a circuit, modulo k, with both sequential and parallel composition read in $\mathbb{Z}_k$. For the qutrit interaction layer we take $g = \mathrm{CNOT}$ and $k = 2$, so qutrit CNOT contributes 1 and every scalar or one-qutrit generator contributes 0. Figure 4 shows the resulting parity check for the binary rule KC: on the bounded 0-, 1-, and 2-wire truncation, the map $\#\{\mathrm{CNOT}\}_{[2]}$ preserves CPh, CZ, and B, but the two sides of KC have opposite parity.

After the one- and two-wire qutrit rules have been separated, the only missing case is the three-wire braid relation I. The open problem is to construct a separating interpretation that remembers enough genuinely ternary entangling structure to distinguish the two sides of (I) while still validating the rest of the reduced calculus, since parity models and other small aritywise targets can detect how many binary entangling gates survive under a translation but not the ordering and interleaving that matter here. Any successful separator must therefore remember genuinely three-wire interaction while still validating the unary and binary layer of the reduced calculus.

IV. Conclusion and Discussion

Completeness tells us that a fragment admits enough equations to derive every valid identity. Minimality asks the harder follow-up: after structural symmetry has moved into the ambient syntax, how many of those equations are still doing indispensable work? Treating symmetry structurally gives a common PROP interface for six near-Clifford fragments, but the payoff is algebraic rather than merely presentational. Formerly explicit controlled and permutation-related laws still have to be recovered from a smaller fragment-specific core, and that core can then be tested axiom by axiom for independence.

Across the six fragments, the outcome has a clear shape. For qubit Clifford, real Clifford, and CNOT-dihedral, the reduced PROP presentations are fully minimal. For bounded-wire Clifford+T, bounded-wire Clifford+CS, and qutrit Clifford, the remaining uncertainty has been compressed to a narrow frontier rather than a diffuse collection of possible redundancies. In the qutrit case, every unary and binary axiom is already known to be necessary, so the open problem has literally been reduced to one ternary braid relation.

This suggests a practical design principle for equational theories of circuit fragments. First isolate the algebraic core. Then test it against small separating models. Only afterwards should one decide which convenience rules to reintroduce for rewriting. Across the family, the separators come from a small recurring toolkit of occurrence maps, counting maps, and a few richer semantic substitutions, so the method is more uniform than the diversity of the fragments might suggest.

The unresolved cases are therefore concrete benchmarks for current independence methods. The bounded-wire fragments need separators that remember one more layer of interaction, while qutrit Clifford needs a genuinely ternary separator for the last braid relation. A successful construction would show what extra semantic data small models must retain, and a negative outcome would still mark a clear boundary for those methods.

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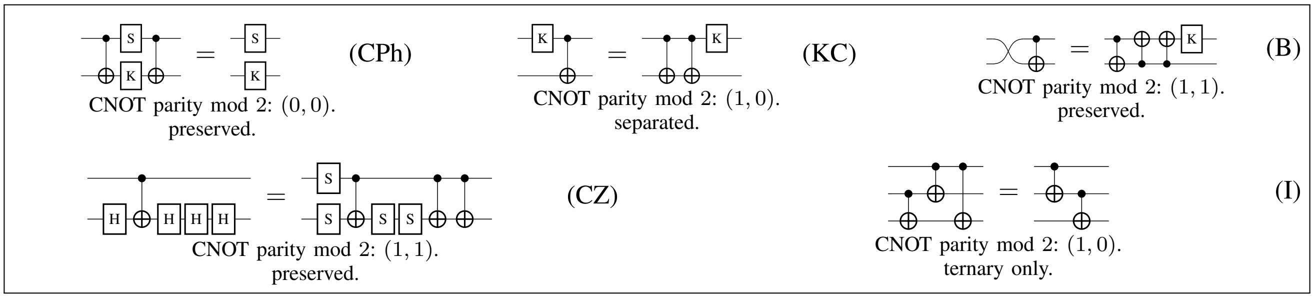

Fig. 4. Worked separator for the qutrit rule KC. Under each panel, "CNOT parity mod 2" gives the value of the counting map $\#\{\mathrm{CNOT}\}_{[2]}$, which counts qutrit CNOT gates modulo 2; each ordered pair is read as $(\text{left}, \text{right})$. The binary rules CPh, CZ, and B are preserved, while KC is distinguished. The ternary braid relation I is included only to indicate the remaining open case beyond binary arity.

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