A Complete and Natural Rule Set for Multi-Qudit Clifford Circuits in All Odd Prime Dimensions

Xiaoning Bian * Sarah Meng Li† Neil J. Ross ‡ John van de Wetering† Yuming Zhao §

Many physical platforms for qubits naturally possess higher energy levels that can encode qudits, enabling greater information density at the cost of increased control complexity [15, 21, 36, 37, 40]. In particular, when the qudit dimension d is an odd prime, qudit systems exhibit rich algebraic structure that result in unique error correction capabilities [13, 39], lower overhead magic state distillation [14, 35], stronger quantum correlations [12, 17], and improvements to various quantum algorithms [9, 33]. Ultimately, there is a trade-off between the engineering cost of accessing higher dimensions and the computational advantages they provide. By characterising qudit circuits in a systematic manner, we can better understand this balance and exploit the potentials of higher-dimensional quantum information processing.

Recently, algebraic approaches to quantum circuits have gained significant attention. An important problem in this line of work is to develop sound and complete equational theories for circuit families, meaning any two circuits representing the same linear map can be transformed into each other using a finite rule set. While there has been substantial progress for qubits [2, 3, 6, 7, 8, 24, 30, 31, 38, 16], characterisation of qudit circuits in terms of generators and relations remains much less explored.

Among quantum circuit fragments, the Clifford group plays a prominent role. Clifford unitaries underpin the stabiliser formalism [1, 19], fault-tolerant quantum computing [22, 23, 25], efficient classical simulation [26, 32], and practical circuit optimisation techniques [10, 11]. As a result, obtaining a complete and finite equational theory for qudit Clifford circuits is a natural and foundational problem.

Building on the characterisation of the qutrit Clifford group [29], where a qutrit is a three-dimensional ($d = 3$) qudit, we present a complete set of rewrite rules for multi-qudit Clifford circuits in all odd prime dimensions. There are 19 *Clifford relations* in total, each involving at most three qudits and admitting an intuitive interpretation.

Theorem. *The rewrite rules in Figure 1 are* complete *for n-qudit Clifford circuits over any qudit dimension d that is an odd prime. That is, given two n-qudit unitary Clifford circuits C_1 and C_2 implementing the same linear map, there is a sequence of rewrites from Figure 1 that proves C_1 and C_2 are equal.*

To prove this, we leverage the isomorphism between the symplectic group $\mathrm{Sp}(2n,\mathbb{Z}_d)$ and the quotient of the Clifford group by the Pauli group. We first derive a complete set of *symplectic relations* for $\mathrm{Sp}(2n,\mathbb{Z}_d)$, and then lift them to Clifford relations by incorporating Pauli corrections. To do this, we introduce a *symplectic normal form* that captures the stabiliser tableau of a Clifford operator. This normal form is then unique for a Clifford unitary up to a Pauli correction. This simplification enables a streamlined derivation of a complete set of 66 parametrised relations, which we further compress to 18 symplectic relations. They suffice to reduce any Clifford circuit to this normal form. Finally, we show how the lifted symplectic relations alongside the Pauli relations can be further reduced to a small set of 19 Clifford relations in Figure 1. In addition to the Clifford generators H, S and CZ, we also use "derived generators" listed in Figure 2, such as SWAP, CX, and the *multiplier* M_g which is defined by its action on the computational basis state, $M_g|x\rangle = |gx\rangle$, where multiplication is over $\mathbb{Z}_d$. These derived generators allow us to present the rewrite rules as intuitive (quasi-)commutation gate relations.

All computations in $\mathrm{Sp}(2n,\mathbb{Z}_d)$ are formalised in the Agda proof assistant, providing a machine-verified proof of correctness. The corresponding code is available in the GitHub repository [5]. Our results not only bridge algebraic group theory with syntactic quantum circuit reasoning, but also lay the foundation for formal verification of quantum protocols in qudit architectures, automated circuit optimisation and synthesis, and the mathematical study of symplectic groups.

*Tsinghua University, Beijing, China
†University of Amsterdam & QuSoft, Amsterdam, Netherlands
‡Dalhousie University, Halifax, Canada
§University of Copenhagen, Copenhagen, Denmark

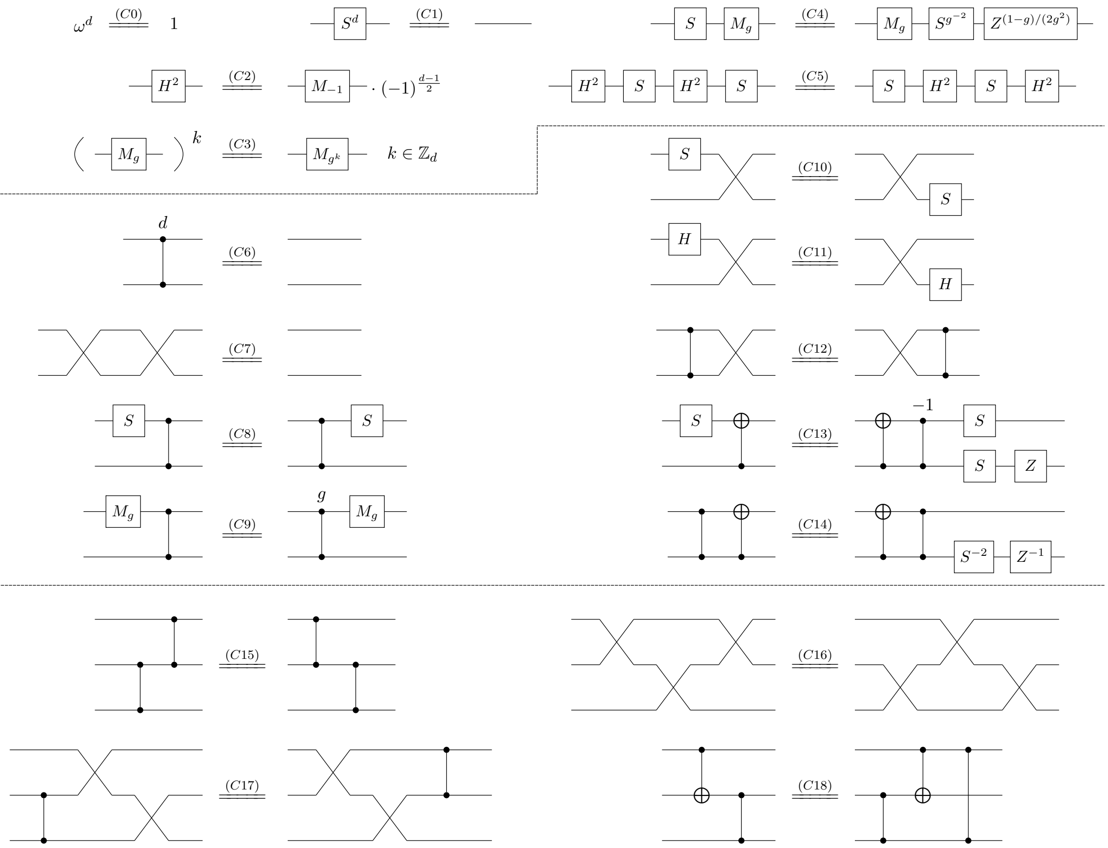


Figure 1: A complete set of rewrite rules for n-qudit Clifford circuits where d is an odd prime and $n \in \mathbb{N}$. Derived generators such as M_g, X, Z, SWAP, and CX are defined in Figure 2.

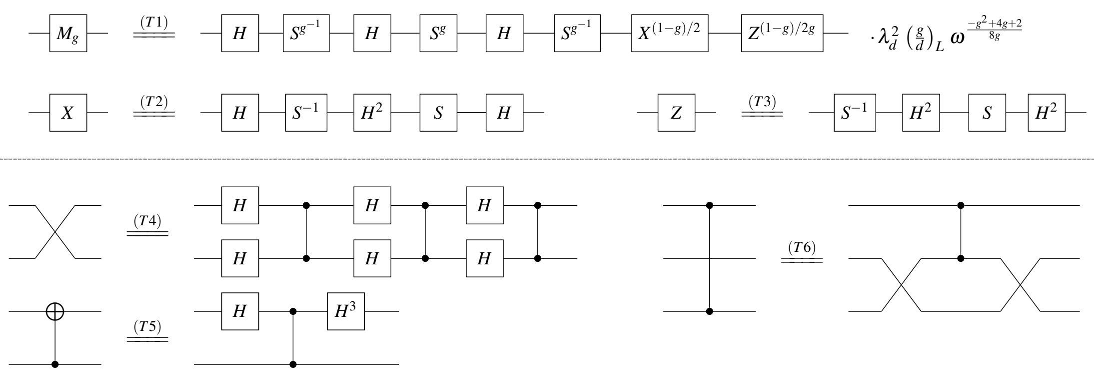


Figure 2: A set of derived Clifford generators, expressed as circuits over the generating set $\{H, S, \text{CZ}\}$. Here, g generates the multiplicative group $\mathbb{Z}_d^*$, and $\left(\frac{g}{d}\right)_L$ denotes the Legendre symbol in number thoery [34, Appendix A].

Preliminaries Let d be an odd prime and let a qudit denote a d-dimensional quantum system. In what follows, we define unitary operators by their actions on the computational basis states. The single-qudit Pauli operators are defined as $X|j\rangle = |j+1\rangle$, $Z|j\rangle = \omega^j|j\rangle$, where $\omega = e^{2\pi i/d}$ and addition is over $\mathbb{Z}_d$. The n-qudit Pauli group $\mathcal{P}_n$ is generated by tensor products of X and Z, together with phases ω^t for $t \in \mathbb{Z}_d$. The qudit H, S, and CZ gates [18, 20, 22, 27, 28, 34] are defined below, where $i = e^{2\pi i/4}$ is the imaginary unit. The choice of global phases ensures that each matrix determinant is equal to 1.

$$H : |j\rangle \mapsto \frac{1}{\lambda_d \sqrt{d}} \sum_{\ell=0}^{d-1} \omega^{j\ell}|\ell\rangle, \;\; \lambda_d = (-i)^{\frac{d-1}{2}}, \qquad S : |j\rangle \mapsto \omega^{\frac{j(j-1)}{2}}|j\rangle, \qquad CZ : |j\rangle|\ell\rangle \mapsto \omega^{j\ell}|j\rangle|\ell\rangle.$$

The n-qudit Clifford group $\mathcal{C}_n$ is the normaliser of $\mathcal{P}_n$ in $\mathcal{U}(d^n)$. It is generated by H, S, and CZ gates via matrix multiplication and tensor product up to global phases that we will ignore. It is well known that the qudit Clifford group modulo Paulis is isomorphic to the symplectic group $\mathrm{Sp}(2n,\mathbb{Z}_d)$ [4]. This correspondence allows Clifford generators to be represented by symplectic matrices over the finite field $\mathbb{Z}_d$, forming the basis of stabiliser tableau representation of Clifford operators [1, 22]. Building on this correspondence, we generalise the prior Clifford completeness results [29, 31, 38] for qubits and qutrits to all odd primes.

A Unique Normal Form for Multi-Qudit Clifford Circuits To find a complete set of rewrite rules, we first define a unique normal form for Clifford circuits. To simplify the construction, we introduce a normal form with two components: a *symplectic normal form* characterising the stabiliser tableau of the Clifford circuit, followed by a *Pauli normal form* giving the necessary Pauli correction. This decomposition reflects the semidirect product structure of the projective Clifford group $\widehat{\mathcal{C}_n}$, where $\widehat{\mathcal{C}_n} \cong \mathrm{Sp}(2n,\mathbb{Z}_d) \ltimes (\mathbb{Z}_d)^{2n}$ [41]. We show that symplectic normal forms $N^{(n)}$ are in one-to-one correspondence with symplectic matrices $M \in \mathrm{Sp}(2n,\mathbb{Z}_d)$. The normal form $N^{(n)}$ is defined inductively using structured layers of circuit fragments, which we call *normal boxes*. These are labelled A, B, D, and E, each implementing a specific action on Pauli generators. An illustration of the multi-qudit Clifford normal form is given in Figure 3.

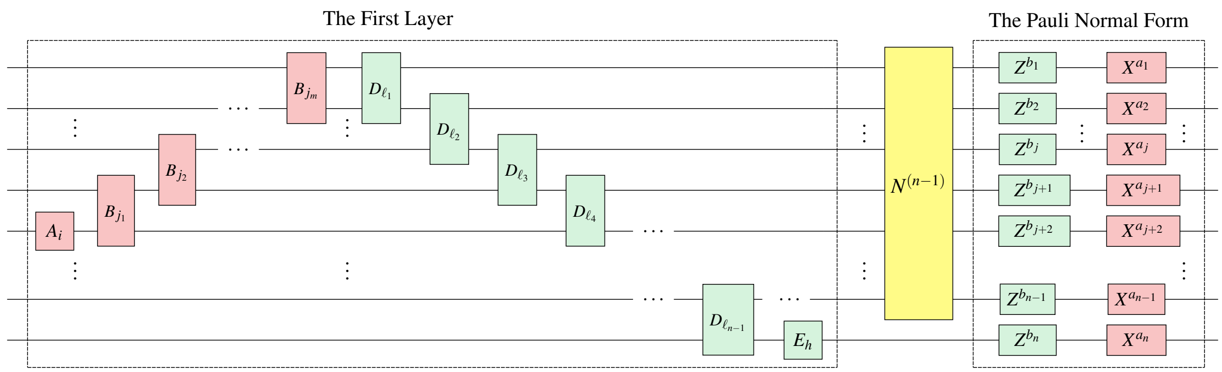
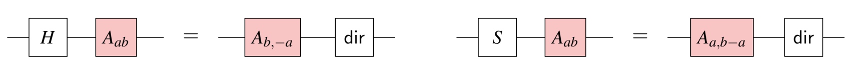


Figure 3: The inductive construction of the normal form for any $C \in \mathcal{C}_n$, up to a global phase. Each layer of the normal form alternates the Z-normal (red) and X-normal (green) circuits, progressively eliminating nontrivial stabiliser tableau entries. The first layer trivialises the conjugation action on the n-th qudit. By induction, the normal form $N^{(n-1)}$ faithfully implements the action of C on the first $n-1$ qudits. Finally, the Pauli normal form carries out the Pauli correction.

Establishing Qudit Clifford Completeness To obtain a complete set of Clifford relations, it suffices to find a set of rewrite rules that transforms any Clifford circuit into its unique normal form. We proceed in three steps. First, we show that if a Clifford generator (H, S, or CZ) is applied before a normal form, the resulting circuit can be rewritten into a new normal form. This is achieved by showing how each generator can be "pushed" through each distinct normal box, yielding equations of the form illustrated below. We refer to them as the *box relations*.



(1)

Here, dir denotes the *residual dirty gates*, which consist of Clifford generators and whose precise form depends on the parameters a, b, and the dimension d. These dirty gates are then pushed further into the remaining part of the normal form. By grouping these relations together in a systematic way, we obtain 66 box relations, each parametrised by the indices of the corresponding normal boxes.

We then reduce these relations to a smaller set of 18 gate relations, forming a compact and natural presentation of the multi-qudit symplectic Clifford group. Finally, we update these relations with Pauli corrections, and show that they imply all the relations governing Paulis and their interaction with Cliffords. This gives Figure 1, a complete set of multi-qudit Clifford relations.

All computations in $\mathrm{Sp}(2n,\mathbb{Z}_d)$ were verified in the proof assistant Agda [5], giving a fully formalised proof of the completeness of the rewrite system. This framework provides a generic method for constructing circuit presentations and enables future verified reasoning about higher-dimensional quantum circuits.

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