Polynomial Spectral Semantics for Magic-State Distillation

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We develop a semantic framework for approximate quantum computation based on spectral transformations of quantum states. We introduce spectral quantum spaces, extending projective quantum state spaces by incorporating effects generated via convex closure of the finite products of commuting effects from the designated set of effects related to the functional calculus in the spectrum of observables. Building on this, we define polynomial spectral signal–state relations, where effect evaluations of parameterized states are polynomial functions of the parameters.

This framework separates structural (spectral) and computational (polynomial) aspects of quantum processes, and provides a natural setting for reasoning about approximation. As a case study, we formalize the Bravyi–Kitaev 5-qubit magic state distillation protocol as a morphism in the category of polynomial spectral signal–state relations. In this formulation, the protocol acts as a rational transformation on spectral parameters, arising from polynomial evaluation followed by normalization.

1 Introduction

Correctness of quantum algorithms is typically expressed via probabilistic guarantees on approximate outputs. A prominent example is magic state distillation [2, 1], where noisy resource states are transformed into higher-fidelity states with (high) probability provided a threshold condition is satisfied. At the same time, many quantum algorithms—most notably those based on quantum singular value transformation [3, 5]—can be understood as implementing polynomial transformations on spectral data of operators. This suggests a unifying perspective: quantum processes can be understood as transformations of spectral values of quantum states. In particular, quantities of interest such as error rates or acceptance probabilities can be expressed as functions of observable spectra.

In this work, we develop a semantic framework for approximate quantum computation based on this idea. We introduce spectral quantum spaces, extending projective quantum state spaces by incorporating effects generated via convex closure of the finite products of the commuting effects from the given set of effects related to the functional calculus in the spectrum of observables. We then define polynomial spectral signal–state relations, where parameterized families of states are related to evaluations of effects that are polynomial in the parameters. This framework separates the structural (spectral) and computational (polynomial) aspects of quantum processes, and provides a natural setting for reasoning about approximation.

As a case study, we show that the Bravyi–Kitaev 5-qubit magic state distillation protocol [2] can be formulated as a morphism in this category followed by normalization.

2 Spectral quantum spaces

We formalize spectral semantics for approximate quantum computation by introducing spectral quantum spaces and polynomial signal-state relation. One could define a projective spectral quantum state space

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(Definition 2.1) as a tuple $(\mathscr{H},\mathscr{P})$ of a finite-dimensional Hilbert space and a set of projection operators. Semantically, the projection operators tell which eigenspace decompositions are observables. In effectus theory [4], they correspond to the predicates which provide information on the state.

Definition 2.1 (Projective spectral quantum state space)**.** A projective spectral quantum state space is a tuple $P = (\mathscr{H},\mathscr{P})$ where $\mathscr{H}$ is a finite-dimensional Hilbert space; and $\mathscr{P}$ is a set of Hermitian idempotents with eigenvalues in $\{0,1\}$.

However, this becomes too rigid for spectral quantum algorithms (which transforms spectral values of quantum states) because the operationally relevant quantity can be obtained by an effect, or convex combination of projection operators. For instance, the twirling (or dephasing) operator from the magic state distillation protocol introduces randomness in the application of operator which can be represented as a convex sum of operators.

We generalize projective spectral quantum state spaces by replacing projection operators with the convex combinations of effects of the products of commuting effects, which are related to the functional calculus on observables (Definition 2.2). Convex combination models classical mixture of effects.

We equip the spectral quantum space X with evaluation map $[\![E]\!]_X(-)$, for all $E \in \mathsf{Eff}_X$, defined by the trace of the product of the effect and the state, which allows us to obtain an effect transformer $\Phi^* : \mathsf{Eff}_Y \to \mathsf{Eff}_X$ for each completely positive trace-non-increasing map $\Phi : \mathsf{St}_X \to \mathsf{St}_Y$ between the states of the spectral quantum space. We can then define the category of spectral quantum space (Definition 2.3) whose objects are spectral quantum spaces and morphisms are completely positive trace-non-increasing map Φ between the quantum states which preserves the effects along the effect transformation Φ^*.

Definition 2.2 (Spectral quantum space)**.** A spectral quantum space is a tuple

$$X = (\mathscr{H},\mathscr{O}_X,\mathsf{Eff}_X,\mathsf{St}_X)$$

where:

- $\mathscr{H}$ is a finite-dimensional Hilbert space.
- $\mathscr{O}_X \subseteq \mathrm{Herm}(\mathscr{H})$ is a distinguished family of observables.
- $\mathsf{Eff}_X \subseteq \mathrm{Eff}(\mathscr{H}) = \{E \mid 0 \le E \le I\}$ is the convex closure of the set $\mathscr{P}_X$ of the products of the commuting effects generated by the set of effects $\mathscr{G}_X$.

  Explicitly, given the set of effects $\mathscr{G}_X$, the set of products of commuting effects is denoted by $\mathscr{P}_X$ as follows:

  $$\mathscr{P}_X = \left\{ \prod_{1\le i\le n} E_i \mid \forall 1 \le i,j \le n.[E_i,E_j] = 0 \text{ and } \forall 1 \le i \le n.E_i \in \mathscr{G}_X \text{ for } n \in \mathbb{N} \right\}$$

  and the convex closure of the set $\mathscr{P}_X$ is denoted by Eff_X as follows:

  $$\mathsf{Eff}_X = \mathsf{Conv}(\mathscr{P}_X)$$

- $\mathsf{St}_X \subseteq \mathscr{D}(\mathscr{H})$ is a convex set of density operators.

The evaluation is defined for each spectral quantum space by

$$[\![E]\!]_X(\rho) = \mathrm{Tr}(E\rho).$$

Definition 2.3 (Category of spectral quantum spaces)**.** The category of spectral quantum spaces is defined with the following data:

- objects: spectral quantum spaces as objects
- morphisms from $X = (\mathscr{H}_X, \mathscr{O}_X, \mathsf{Eff}_X, \mathsf{St}_X)$ to $Y = (\mathscr{H}_Y, \mathscr{O}_Y, \mathsf{Eff}_Y, \mathsf{St}_Y)$: completely positive trace-non-increasing (CP TNI) map $\Phi : \mathsf{St}_X \to \mathsf{St}_Y$ such that $\Phi^*(E_Y) \in \mathsf{Eff}_X$ for all $E_Y \in \mathsf{Eff}_Y$ where Φ^* Heisenberg dual defined by

$$[\![E_Y]\!]_Y(\Phi(\rho)) = \mathrm{Tr}(E_Y \Phi(\rho)) = \mathrm{Tr}(\Phi^*(E_Y)\rho) = [\![\Phi^*(E_Y)]\!]_X(\rho).$$

  Explicitly, when $\Phi(\rho) = \sum_{i=1}^n K_i \rho K_i^\dagger$ for Kraus operators $\{K_i\}$ then, $\Phi^*(E) = \sum_{i=1}^n K_i^\dagger E K_i$.

3 Polynomial spectral signal-state relation

Effects of the spectral quantum space allow us to inspect properties of the states. Conversely, we may have knowledge on the states. It is formalized in the spectral signal-state relation (Definition 3.1) where we deals with the parameterized family of states whose parameter forms a semialgebraic set of the parameter space. Composing the parameterized family of states with the effects through the evaluation function, we obtain a function $e_E : M \to [0,1]$ for each effect $E \in \mathsf{Eff}_X$. We call the spectral signal-state relation polynomial when the composition e_E is polynomial for all $E \in \mathsf{Eff}_X$.

Note that we require polynomiality at the level of observable evaluation rather than state representation. This makes the definition representation-independent and aligns with operational semantics where states are characterized by their observable statistics.

Definition 3.1 (Polynomial spectral signal-state relation)**.** A spectral signal-state relation is defined by a triple (M, X, ρ) where M refers to parameter space (i.e. a semialgebraic set of parameters), X refers to a spectral quantum space from Definition 2.2, and the map $\rho : M \to \mathsf{St}_X$ refers to a relation between the signal and the state of the spectral quantum space. When the function $e_E : M \to [0,1]$ defined by

$$e_E : m \mapsto [\![E]\!]_X(\rho(m)) = \mathrm{Tr}(E\rho(m))$$

is a polynomial for every effect $E \in \mathsf{Eff}_X$, the signal-state relation (M, X, ρ) is called polynomial spectral signal-state relation.

Definition 3.2 (Category of polynomial spectral signal-state relation)**.** Category of polynomial spectral signal-state relation is defined with the following data:

- objects: polynomial spectral signal-state relations (M, X, ρ)
- morphisms $(M, X, \rho) \to (N, Y, \psi)$: pair of morphisms (f, Φ) of a morphism f in the signal spaces and a morphism Φ in the category of spectral quantum spaces such that the following equality holds:

$$[\![\Phi^*(E_Y)]\!]_X(\rho(m)) = \mathrm{Tr}(\Phi^*(E_Y)\rho(m)) = \mathrm{Tr}(E_Y \psi(f(m))) = [\![E_Y]\!]_Y(\psi(f(m)))$$

  for all $E_Y \in \mathsf{Eff}_Y$ and all $m \in M$.

Remark. This structure suggests a Grothendieck construction over parameter spaces, which we leave for future work.

4 Magic state distillation protocol

Let us formulate the 5-qubit magic state distillation protocol by Bravyi and Kitaev [2] (a summary of which can be found in the Appendices) in the context of polynomial spectral signal-state relation. We will divide the protocol into two parts and formulate each part as morphism in the category of polynomial spectral signal-state relation. First, let us specify the spectral quantum space and the polynomial spectral signal-state relation at each step around the two parts:

- initial state:
  - spectral quantum space: $X_0 = (\mathscr{H}_2^{\otimes 5},\ \mathscr{O}_0,\ \mathsf{Eff}_0,\ \mathsf{Conv}(\mathsf{Img}(\rho_0)))$ where $\mathsf{Img}(\rho_0)$ refers to the image of the parameterized family of states, $\mathsf{Conv}(-)$ refers to the convex closure of given set, and
  $$\mathscr{O}_0 = \left\{ \frac{1}{3}\left(H + T^\dagger H T + T H T^\dagger\right) \mid H \in \mathscr{O}_1 \right\}$$
  and $\mathsf{Eff}_0 = \mathsf{Conv}(\mathscr{P}_0)$ is the convex closure of the set of products of commuting effects of the following generating set of effects
  $$\mathscr{G}_0 = \left\{ \frac{1}{3}\left(E + T^\dagger E T + T E T^\dagger\right) \mid E \in \mathscr{P}_1 \right\}$$
  - polynomial spectral signal-state relation:
  $$R_0 = \left(M_0 = \{(\alpha,\beta,\gamma) \in \mathbb{R}^3 \mid \alpha^2+\beta^2+\gamma^2 \le 1\},\ X_0,\ \rho_0 : (\alpha,\beta,\gamma) \mapsto \left(\frac{1}{2}(I + \alpha X + \beta Y + \gamma Z)\right)^{\otimes 5} \right)$$
- after dephasing channel:
  - spectral quantum space: $X_1 = (\mathscr{H}_2^{\otimes 5},\ \mathscr{O}_1,\ \mathsf{Eff}_1,\ \mathsf{Conv}(\mathsf{Img}(\rho_1)))$ where the observables are given by the stabilizers $\mathscr{S} = \{S_1, S_2, S_3, S_4\}$ of the 5-qubit code and the observable for the logical qubit
  $$\mathscr{O}_1 = \mathscr{S} \cup \{\left|T_0^L\right\rangle \left\langle T_0^L\right| - \left|T_1^L\right\rangle \left\langle T_1^L\right|\}$$
  and $\mathsf{Eff}_1 = \mathsf{Conv}(\mathscr{P}_1)$ is the convex closure of the products
  $$\mathscr{P}_1 = \left\{ \prod_{i=1}^{4} \left(P_{S_i,+}^{\frac{(\alpha_i+1)^2-1}{8}} P_{S_i,-}^{\frac{(\alpha_i-1)^2-1}{8}} \right) \mid \alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \{-2,0,2\} \right\} \cup \{\left|T_0^L\right\rangle \left\langle T_0^L\right|, \left|T_1^L\right\rangle \left\langle T_1^L\right|\}$$
  of commuting effects from the following generating set of effects
  $$\mathscr{G}_1 = \left\{ P_{S,+1} = \frac{I^{\otimes 5} + S}{2},\ P_{S,-1} = \frac{I^{\otimes 5} - S}{2} \mid S \in \mathscr{O}_1 \right\} \cup \{\left|T_0^L\right\rangle \left\langle T_0^L\right|, \left|T_1^L\right\rangle \left\langle T_1^L\right|\}$$
  where
  $$\left|T_0^L\right\rangle = U_{\text{dec}}^\dagger(|T_1\rangle \otimes |0000\rangle) \qquad \left|T_1^L\right\rangle = U_{\text{dec}}^\dagger(|T_0\rangle \otimes |0000\rangle)$$
  - polynomial spectral signal-state relation:
  $$R_1 = \left(M_1 = [0,1],\ X_1,\ \rho_1 : \varepsilon \mapsto ((1-\varepsilon)\,|T_0\rangle\langle T_0| + \varepsilon\,|T_1\rangle\langle T_1|)^{\otimes 5} \right)$$
- after projection and postselection:

– spectral quantum space: $X_2 = (\mathscr{H}_2,\ \mathscr{O}_2,\ \mathsf{Eff}_2,\ \mathsf{Conv}(\mathsf{Img}(\rho_2)))$ where $\mathscr{O}_2 = \{|T_0\rangle\langle T_0| - |T_1\rangle\langle T_1|\}$ and $\mathsf{Eff}_2 = \mathsf{Conv}(\mathscr{P}_2)$ is the convex closure of the set of products

$$\mathscr{P}_2 = \{0,\ |T_0\rangle\langle T_0|,\ |T_1\rangle\langle T_1|,\ I\}$$

of commuting effects from $\mathscr{G}_2 = \{|T_0\rangle\langle T_0|, |T_1\rangle\langle T_1|\}$

– polynomial spectral signal-state relation:

$$R_2 = (M_2 = [0,1]^2,\ X_2,\ \rho_2 : (\mu,\varepsilon) \mapsto \mu(1-\varepsilon)\,|T_0\rangle\langle T_0| + \mu\varepsilon\,|T_1\rangle\langle T_1|)$$

Note that the effects $P_{S,+1}$ and $P_{S,-1}$ in Eff_1 are projections onto the $+1$ eigenspace and -1 eigenspace, respectively. In particular, the convex set of effects Eff_1 contains the product of projections, which corresponds to the projection onto the codespace, namely E_{acc}. Also, we define G_0 as the image of the commuting effect algebra P_1 under the Heisenberg dual Φ_1^*, ensuring closure of effects under the transformation.

The two parts of the protocol can be formalized as morphisms in the category of polynomial spectral signal-state relations:

- Dephasing (Twirling): The first morphism is given by the completely positive trace-preserving map

$$\Phi_1(\rho) = D^{\otimes 5}(\rho),$$

together with the signal map

$$f_1 : (\alpha,\beta,\gamma) \mapsto \mathrm{Tr}\left(|T_1\rangle\langle T_1|\, D\left(\frac{1}{2}(I + \alpha X + \beta Y + \gamma Z)\right)\right),$$

which extracts the error parameter ε of the twirled state.

- Projection and postselection: Let E_{acc} be the projector onto the code space. Let U_{dec} be a decoding Clifford circuit mapping the code space to $\mathscr{H}_2 \otimes |0000\rangle$. The accepted branch is then given by the completely positive trace-non-increasing map

$$\Phi_2(\rho) = (I \otimes \langle 0000|)\, U_{\text{dec}}\, \mathsf{E}_{\text{acc}}\, \rho\, \mathsf{E}_{\text{acc}}\, U_{\text{dec}}^{\dagger}\, (I \otimes |0000\rangle)$$

whose acceptance probability is $p_{\text{acc}}(\varepsilon) = \mathrm{Tr}(E_{\text{acc}}\rho_1(\varepsilon))$ and the error is $\varepsilon_{\text{out}}(\varepsilon) = \mathrm{Tr}(|T_1\rangle\langle T_1|\,\Phi_2(\rho_1(\varepsilon)))$. The output parameter is given by

$$f_2 : \varepsilon \mapsto (p_{\text{acc}}(\varepsilon), \varepsilon_{\text{out}}(\varepsilon)),$$

Proposition 4.1. *The Bravyi–Kitaev* 5*-qubit protocol defines a polynomial spectral signal–state morphism.*

This shows that the magic state distillation protocol acts as a rational transformation on spectral parameters, arising from a polynomial evaluation followed by normalization.

5 Discussion

We have shown that the Bravyi–Kitaev 5-qubit magic state distillation protocol admits a natural formulation in polynomial spectral semantics. In this framework, the protocol is expressed as a composition of morphisms acting on spectral parameters, where the acceptance probability is polynomial and the output error is obtained as a rational function via normalization.

This perspective aligns closely with quantum singular value transformation, where quantum circuits implement polynomial transformations of spectral data. Our framework suggests that magic state distillation and related fault-tolerant procedures can be understood uniformly as spectral transformations.

This framework connects quantum error correction, polynomial approximation, and categorical quantum theory. An interesting direction for future work is to extend this framework to more general code families, such as Reed–Muller and topological codes, and to characterize the class of achievable spectral transformations.

Another direction is to relax the exact polynomial condition and study approximate signal–state relations, which may provide a semantic account of noise and robustness in quantum algorithms.

Finally, the categorical structure suggests a Grothendieck construction over parameter spaces, which would lead to more abstract categorical model for spectral quantum processes.

References

[1] Sergey Bravyi & Jeongwan Haah (2012): *Magic-state distillation with low overhead*. Physical Review A.

[2] Sergey Bravyi & Alexei Kitaev (2005): *Universal quantum computation with ideal Clifford gates and noisy ancillas*. Physical Review A 71(2).

[3] András Gilyén, Yuan Su, Guang Hao Low & Nathan Wiebe (2019): *Quantum singular value transformation and beyond*. STOC.

[4] Bart Jacobs (2015): *New directions in categorical logic, for classical, probabilistic and quantum logic*. Logical Methods in Computer Science.

[5] Guang Hao Low & Isaac L. Chuang (2017): *Optimal Hamiltonian simulation by quantum signal processing*. PRL.

Appendices

Bravyi and Kitaev's 5-qubit magic state distillation protocol

Bravyi and Kitaev's 5-qubit magic state distillation protocol [2] takes 5 qubits ($\rho^{\otimes 5}$) each of which is represented, after applying the dephasing channel $D : \mathscr{H}_2 \to \mathscr{H}_2$, as

$$D(\rho) = \frac{1}{3}(\rho + T\rho T^\dagger + T^\dagger \rho T)$$

where $T = SH = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ and $|T_{0,1}\rangle\langle T_{0,1}| = \frac{1}{2}\left(I \pm \frac{1}{\sqrt{3}}(X+Y+Z)\right)$ where X, Y, and Z refer to the Pauli operators

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and S and H refer to the following operators

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \qquad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Explicitly,

$$|T_0\rangle = \cos\beta\,|0\rangle + e^{\frac{i\pi}{4}}\sin\beta\,|1\rangle \qquad |T_1\rangle = e^{\frac{i\pi}{4}}\sin\beta\,|0\rangle - \cos\beta\,|1\rangle$$

where $\cos(2\beta) = \frac{1}{\sqrt{3}}$. Note that $|T_0\rangle$ and $|T_1\rangle$ are eigenvectros of T (i.e. $T\,|T_0\rangle = e^{\frac{i\pi}{3}}\,|T_0\rangle$ and $T\,|T_1\rangle = e^{-\frac{i\pi}{3}}\,|T_1\rangle$) with regard to the eigenvalues $e^{\frac{i\pi}{3}}$ and $e^{-\frac{i\pi}{3}}$, respectively, and

$$\begin{aligned}
D(|T_0\rangle\langle T_0|) &= \frac{1}{3}\left(|T_0\rangle\langle T_0| + |T_0\rangle\langle T_0| + |T_0\rangle\langle T_0|\right) = |T_0\rangle\langle T_0| \\
D(|T_0\rangle\langle T_1|) &= \frac{1}{3}\left(|T_0\rangle\langle T_1| + e^{\frac{2i\pi}{3}}\,|T_0\rangle\langle T_1| + e^{-\frac{2i\pi}{3}}\,|T_0\rangle\langle T_1|\right) = 0 \\
D(|T_1\rangle\langle T_0|) &= \frac{1}{3}\left(|T_1\rangle\langle T_0| + e^{-\frac{2i\pi}{3}}\,|T_1\rangle\langle T_0| + e^{\frac{2i\pi}{3}}\,|T_1\rangle\langle T_0|\right) = 0 \\
D(|T_1\rangle\langle T_1|) &= \frac{1}{3}\left(|T_1\rangle\langle T_1| + |T_1\rangle\langle T_1| + |T_1\rangle\langle T_1|\right) = |T_1\rangle\langle T_1|
\end{aligned}$$

Since the dephasing operator forgets the non-diagonal elements of the density operator, we can represent it (the state obtained by applying the dephasing channel) as erroneous T-state with error ε as follows:

$$D(\rho) = (1-\varepsilon)\,|T_0\rangle\langle T_0| + \varepsilon\,|T_1\rangle\langle T_1|$$

Given this representation, the five separately prepared erroneous T-state with error ε can be represented as follows:

$$D(\rho)^{\otimes 5} = \left((1-\varepsilon)\,|T_0\rangle\langle T_0| + \varepsilon\,|T_1\rangle\langle T_1|\right)^{\otimes 5} = \sum_{x\in\{0,1\}^5} (1-\varepsilon)^{5-|x|}\varepsilon^{|x|}\,|T_x\rangle\langle T_x|$$

where $|T_x\rangle = |T_{x_1}\rangle \otimes \cdots \otimes |T_{x_5}\rangle$ for $x = (x_1, \ldots, x_5) \in \{0,1\}^5$.

Next, these erroneous states $D(\rho)^{\otimes 5}$ is projected onto the codespace of the 5-qubit code (consists of the stabilizers $\mathscr{S} = \{S_1 = XZZXI,\ S_2 = IXZZX,\ S_3 = XIXZZ,\ S_4 = ZXIXZ\}$) which is implemented by the decoding algorithm (a unitary map V such that $V S_j V^\dagger = I_2^j \otimes Z \otimes I_2^{4-j}$ for all $S_j \in \mathscr{S}$) for the 5-qubit code followed by 4 Z-basis measurements and an operator $A = YH = \frac{1}{\sqrt{2}}\begin{pmatrix} -i & i \\ i & i \end{pmatrix}$ that swaps the states $|T_0\rangle$ and $|T_1\rangle$ (i.e. $|T_1\rangle = A\,|T_0\rangle$ and $|T_0\rangle = A\,|T_1\rangle$). The protocol returns the purified magic state $((1-\varepsilon_{\text{out}})(\varepsilon)\,|T_0\rangle\langle T_0| + \varepsilon_{\text{out}}(\varepsilon)\,|T_1\rangle\langle T_1|)$ when the measurement outcomes of the Z-basis measurements are all $+1$.

Specifically, in [2], the codespace is shown to be spanned by the basis states $\left|T_1^L\right\rangle = \sqrt{6}\mathsf{E}_{\text{acc}}\,|T_{00000}\rangle$ and $\left|T_0^L\right\rangle = \sqrt{6}\mathsf{E}_{\text{acc}}\,|T_{11111}\rangle$ where E_{acc} is the projection operator onto the codespace defined by:

$$\mathsf{E}_{\text{acc}} = \prod_{S\in\mathscr{S}} P_{S,+1} = \prod_{S\in\mathscr{S}} \frac{I^{\otimes 5}+S}{2} = \frac{1}{2^4}\sum_{b_1,b_2,b_3,b_4\in\{0,1\}} S_1^{b_1}S_2^{b_2}S_3^{b_3}S_4^{b_4} = \frac{1}{16}\sum_{g\in\langle\mathscr{S}\rangle} g$$

The equality follows from the fact that the stabilizer generators commute ($S_iS_j = S_jS_i$ for $i \neq j$) and square to the identity ($S_i^2 = I$). More precisely, we can show that $\left|T_0^L\right\rangle$ and $\left|T_1^L\right\rangle$ are eigenvectors of the operator $\hat{T} = T^{\otimes 5}$ with regard to the eigenvalues $e^{\frac{i\pi}{3}}$ and $e^{-\frac{i\pi}{3}}$ as follows:

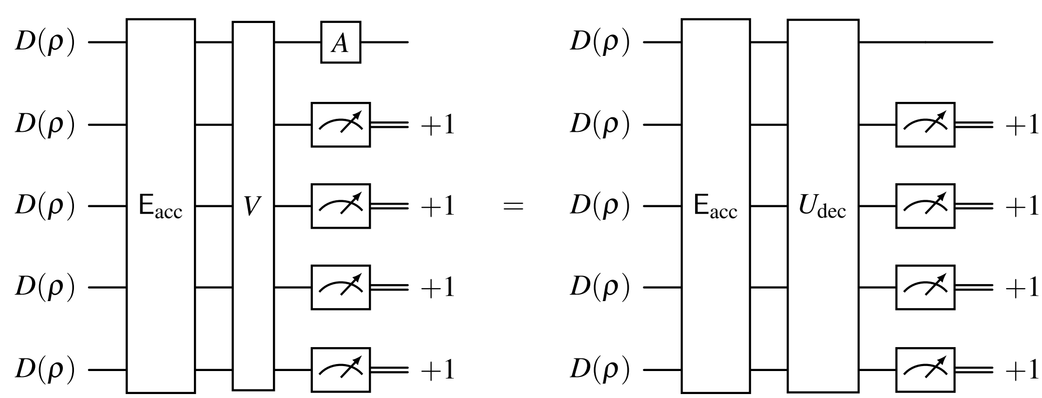
$$\hat{T}\left|T_1^L\right\rangle = \sqrt{6}\hat{T}\mathsf{E}_{\text{acc}}\left|T_{00000}\right\rangle = \sqrt{6}\mathsf{E}_{\text{acc}}\hat{T}\left|T_{00000}\right\rangle = \sqrt{6}e^{\frac{5i\pi}{3}}\mathsf{E}_{\text{acc}}\left|T_{00000}\right\rangle = \sqrt{6}e^{-\frac{i\pi}{3}}\left|T_1^L\right\rangle$$
$$\hat{T}\left|T_0^L\right\rangle = \sqrt{6}\hat{T}\mathsf{E}_{\text{acc}}\left|T_{11111}\right\rangle = \sqrt{6}\mathsf{E}_{\text{acc}}\hat{T}\left|T_{11111}\right\rangle = \sqrt{6}e^{-\frac{5i\pi}{3}}\mathsf{E}_{\text{acc}}\left|T_{11111}\right\rangle = \sqrt{6}e^{\frac{i\pi}{3}}\left|T_0^L\right\rangle$$

by using the fact that $\hat{T}\mathsf{E}_{\text{acc}} = \mathsf{E}_{\text{acc}}\hat{T}$ and the fact that $|T_0\rangle$ and $|T_1\rangle$ are eigenvectors of the operator T. It follows that $\left|T_0^L\right\rangle$ and $\left|T_1^L\right\rangle$ are orthogonal (since they are eigenvectors of unitary operator $\hat{T}$ corresponding to distinct eigenvalues) and we can also show that they are normalized as

$$\begin{aligned}\left\langle T_1^L\right|\left|T_1^L\right\rangle &= 6\left\langle T_{00000}\right|\mathsf{E}_{\text{acc}}^2\left|T_{00000}\right\rangle = 6\mathrm{Tr}(\left\langle T_{00000}\right|\mathsf{E}_{\text{acc}}\left|T_{00000}\right\rangle) = 6\mathrm{Tr}(\mathsf{E}_{\text{acc}}\left|T_{00000}\right\rangle\left\langle T_{00000}\right|)\\ &= 6\mathrm{Tr}\left(\mathsf{E}_{\text{acc}}\left(\frac{1}{2}(I+\frac{1}{\sqrt{3}}(X+Y+Z))\right)^{\otimes 5}\right) = \frac{6}{2^5 2^4}\sum_{g\in\langle\mathscr{S}\rangle}\mathrm{Tr}\left(g\left(I+\frac{1}{\sqrt{3}}(X+Y+Z)\right)^{\otimes 5}\right)\\ &= \frac{3}{2^8}\sum_{g\in\langle\mathscr{S}\rangle}\sum_{h\in\mathscr{S}_+(5)}\mathrm{Tr}\left(\frac{gh}{\sqrt{3}^{|h|}}\right) = \frac{3}{2^8}\sum_{g\in\langle\mathscr{S}\rangle}\left(\frac{2^5}{\sqrt{3}^{|g|}}\right) = \frac{3}{8}\sum_{g\in\langle\mathscr{S}\rangle}3^{-\frac{|g|}{2}} = \frac{3}{8}(1+\frac{15}{9}) = 1\end{aligned}$$

and, similarly, $\left\langle T_0^L\right|\left|T_0^L\right\rangle = 1$ where $\mathscr{S}_+(5) = \{I,X,Y,Z\}^{\otimes 5}$ and $|h|$ refers to the number of non-trivial parts in h. Note that the group $\langle\mathscr{S}\rangle$ generated by the stabilizers $\mathscr{S}$ consists of one identity operator and 15 operators with 4 non-trivial parts.

Given the basis $\left|T_0^L\right\rangle$ and $\left|T_1^L\right\rangle$ of the codespace, the decoding algorithm V followed by the swap operator A can be produces an operator $U_{\text{dec}} : \mathscr{H}_2^{\otimes 5} \to \mathscr{H}_2^{\otimes 5}$ such that



and

$$U_{\text{dec}}\left|T_0^L\right\rangle = |T_1\rangle\otimes|0000\rangle \qquad U_{\text{dec}}\left|T_1^L\right\rangle = |T_0\rangle\otimes|0000\rangle$$

Since U_{dec} is unitary, it follows that

$$\left|T_0^L\right\rangle = U_{\text{dec}}^\dagger(|T_1\rangle\otimes|0000\rangle) \qquad \left|T_1^L\right\rangle = U_{\text{dec}}^\dagger(|T_0\rangle\otimes|0000\rangle)$$

Also, for $x \in \{0,1\}^5$, since E_{acc} commutes with $\hat{T}$,

$$\hat{T}\mathsf{E}_{\text{acc}}|T_x\rangle = \mathsf{E}_{\text{acc}}\hat{T}|T_x\rangle = \mathsf{E}_{\text{acc}}e^{\frac{i\pi(5-2|x|)}{3}}|T_x\rangle = \begin{cases} e^{-\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle, & |x|=0 \\ e^{i\pi}\mathsf{E}_{\text{acc}}|T_x\rangle, & |x|=1 \\ e^{\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle, & |x|=2 \\ e^{-\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle, & |x|=3 \\ e^{-i\pi}\mathsf{E}_{\text{acc}}|T_x\rangle, & |x|=4 \\ e^{\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle, & |x|=5 \end{cases}$$

It follows that $\mathsf{E}_{\text{acc}}|T_x\rangle$s are eigenvectors of $\hat{T}$ for all $x \in \{0,1\}^5$. Moreover, since E_{acc} is a projector onto the subspace spanned by $|T_0^L\rangle$ and $|T_1^L\rangle$, and the eigenvectors with distinct eigenvalues are orthogonal, we can infer that

$$\begin{cases} e^{-\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle = e^{-\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_{00000}\rangle = \frac{1}{\sqrt{6}}\hat{T}|T_1^L\rangle, & |x|=0 \\ e^{i\pi}\mathsf{E}_{\text{acc}}|T_x\rangle = 0, & |x|=1 \\ e^{\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle = a_x\hat{T}|T_0^L\rangle, & |x|=2 \\ e^{-\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle = b_x\hat{T}|T_1^L\rangle, & |x|=3 \\ e^{-i\pi}\mathsf{E}_{\text{acc}}|T_x\rangle = 0, & |x|=4 \\ e^{\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_x\rangle = e^{\frac{i\pi}{3}}\mathsf{E}_{\text{acc}}|T_{11111}\rangle = \frac{1}{\sqrt{6}}\hat{T}|T_0^L\rangle, & |x|=5 \end{cases}$$

with some constants a_xs, for $|x|=2$, and b_xs, for $|x|=3$. However, since $I = \sum_{x\in\{0,1\}^5}|T_x\rangle\langle T_x|$, it follows that

$$\begin{aligned} |T_0\rangle\langle T_0| + |T_1\rangle\langle T_1| = \mathsf{E}_{\text{acc}} = \mathsf{E}_{\text{acc}}I\mathsf{E}_{\text{acc}} &= \sum_{x\in\{0,1\}^5}\mathsf{E}_{\text{acc}}|T_x\rangle\langle T_x|\mathsf{E}_{\text{acc}} \\ &= \left(\frac{1}{6} + \sum_{|x|=3}|b_x|^2\right)|T_1^L\rangle\langle T_1^L| + \left(\frac{1}{6} + \sum_{|x|=2}|a_x|^2\right)|T_0^L\rangle\langle T_0^L| \end{aligned}$$

It follows from the equation that $\sum_{|x|=3}|b_x|^2 = \frac{5}{6}$ and $\sum_{|x|=2}|a_x|^2 = \frac{5}{6}$. Therefore, after applying the projection E_{acc} onto the codespace unto the state $D(\rho)^{\otimes 5}$, we obtain the following state:

$$\begin{aligned} \mathsf{E}_{\text{acc}}D(\rho)^{\otimes 5}\mathsf{E}_{\text{acc}} &= \sum_{x\in\{0,1\}^5}(1-\varepsilon)^{5-|x|}\varepsilon^{|x|}\mathsf{E}_{\text{acc}}|T_x\rangle\langle T_x|\mathsf{E}_{\text{acc}} \\ &= \left(\frac{(1-\varepsilon)^5}{6} + \frac{5(1-\varepsilon)^2\varepsilon^3}{6}\right)|T_1^L\rangle\langle T_1^L| + \left(\frac{5(1-\varepsilon)^3\varepsilon^2}{6} + \frac{\varepsilon^5}{6}\right)|T_0^L\rangle\langle T_0^L| \end{aligned}$$

hence

$$U_{\text{dec}}\mathsf{E}_{\text{acc}}D(\rho)^{\otimes 5}\mathsf{E}_{\text{acc}}U_{\text{dec}}^\dagger = \left(\left(\frac{(1-\varepsilon)^5}{6} + \frac{5(1-\varepsilon)^2\varepsilon^3}{6}\right)|T_0\rangle\langle T_0| + \left(\frac{5(1-\varepsilon)^3\varepsilon^2}{6} + \frac{\varepsilon^5}{6}\right)|T_1\rangle\langle T_1|\right)\otimes|0000\rangle\langle 0000|$$

By letting $p_{\text{acc}}(\varepsilon)$ be the probability of observing the trivial syndrome is obtained and $\varepsilon_{\text{out}}(\varepsilon)$ be the error in the distilled state, we can rewrite the equation as follows:

$$U_{\text{dec}}\mathsf{E}_{\text{acc}}D(\rho)^{\otimes 5}\mathsf{E}_{\text{acc}}U_{\text{dec}}^\dagger = (p_{\text{acc}}(\varepsilon)(1-\varepsilon_{\text{out}}(\varepsilon))|T_0\rangle\langle T_0| + p_{\text{acc}}(\varepsilon)\varepsilon_{\text{out}}(\varepsilon)|T_1\rangle\langle T_1|)\otimes|0000\rangle\langle 0000|$$

where

$$p_{\mathrm{acc}}(\varepsilon) = \frac{(1-\varepsilon)^5 + 5\varepsilon^2(1-\varepsilon)^3 + 5\varepsilon^3(1-\varepsilon)^2 + \varepsilon^5}{6}.$$

and

$$\varepsilon_{\mathrm{out}}(\varepsilon) = \frac{\varepsilon^5 + 5\varepsilon^2(1-\varepsilon)^3}{(1-\varepsilon)^5 + 5\varepsilon^2(1-\varepsilon)^3 + 5\varepsilon^3(1-\varepsilon)^2 + \varepsilon^5}.$$

Proof of Proposition 4.1

Proof. We first show that X_0, X_1, and X_2 are valid spectral quantum spaces, and that Φ_1 and Φ_2 are valid spectral quantum space morphisms. Recall that

$$\begin{aligned} X_0 &= (\mathscr{H}_2^{\otimes 5}, \mathscr{O}_0, \mathsf{Eff}_0, \mathsf{Conv}(\mathsf{Img}(\rho_0))) \\ X_1 &= (\mathscr{H}_2^{\otimes 5}, \mathscr{O}_1, \mathsf{Eff}_1, \mathsf{Conv}(\mathsf{Img}(\rho_1))) \\ X_2 &= (\mathscr{H}_2, \mathscr{O}_2, \mathsf{Eff}_2, \mathsf{Conv}(\mathsf{Img}(\rho_2))) \end{aligned}$$

where

$$\begin{aligned} \mathscr{O}_0 &= \left\{ \frac{1}{3^5} \sum_{\alpha \in \{-1,0,1\}^5} \left(T^{\otimes\alpha}\right)^\dagger H T^{\otimes\alpha} \mid H \in \mathscr{O}_1 \right\} \\ \mathscr{O}_1 &= \{XZZXI, IXZZX, XIXZZ, ZXIXZ\} \cup \{\left|T_0^L\right\rangle\left\langle T_0^L\right| - \left|T_1^L\right\rangle\left\langle T_1^L\right|\} \\ \mathscr{O}_2 &= \{|T_0\rangle\langle T_0| - |T_1\rangle\langle T_1|\} \end{aligned}$$

and Eff_i (for $i = 0,1,2$) is the convex closure of the set of finite products $\mathscr{P}_i$ of commuting effects generated by the following sets of effects $\mathscr{G}_i$, respectively:

- $\mathscr{G}_0 = \left\{ \frac{1}{3^5} \sum_{\alpha\in\{-1,0,1\}^5} (T^{\otimes\alpha})^\dagger E T^{\otimes\alpha} \mid E \in \mathscr{P}_1 \right\}$
- $\mathscr{G}_1 = \left\{ P_{S,+1} = \frac{I^{\otimes 5}+S}{2},\ P_{S,-1} = \frac{I^{\otimes 5}-S}{2} \mid S \in \mathscr{O}_1 \right\} \cup \{\left|T_0^L\right\rangle\left\langle T_0^L\right|,\ \left|T_1^L\right\rangle\left\langle T_1^L\right|\}$
- $\mathscr{G}_2 = \{|T_0\rangle\langle T_0|,\ |T_1\rangle\langle T_1|\}$

First, note that, by definition, the states $\mathsf{Conv}(\mathsf{Img}(\rho_0))$, $\mathsf{Conv}(\mathsf{Img}(\rho_1))$, and $\mathsf{Conv}(\mathsf{Img}(\rho_2))$ are convex sets of density operators. Next, we need to show that:

- The quantum channels Φ_1 and Φ_2 send, respectively, each element of $\mathsf{Conv}(\mathsf{Img}(\rho_0))$ to an element of $\mathsf{Conv}(\mathsf{Img}(\rho_1))$ and each element of $\mathsf{Conv}(\mathsf{Img}(\rho_1))$ to an element of $\mathsf{Conv}(\mathsf{Img}(\rho_2))$. Note that

  $$\Phi_1(\rho^{\otimes 5}) = D^{\otimes 5}(\rho^{\otimes 5}) = \frac{1}{3^5}\left(\rho + T\rho T^\dagger + T^\dagger \rho T\right)^{\otimes 5}$$

  and

  $$\Phi_2(\rho) = (I \otimes \langle 0000|)\, U_{\mathrm{dec}}\, \mathsf{E}_{\mathrm{acc}}\, \rho\, \mathsf{E}_{\mathrm{acc}}\, U_{\mathrm{dec}}^\dagger\, (I \otimes |0000\rangle)$$

  For any (α,β,γ) such that $\alpha^2+\beta^2+\gamma^2 \leq 1$,

  $$\begin{aligned} \Phi_1(\rho_0(\alpha,\beta,\gamma)) &= D^{\otimes 5}\left(\frac{1}{2}(I + \alpha X + \beta Y + \gamma Z)\right)^{\otimes 5} \\ &= \left(D\left(\frac{1}{2}(I + \alpha X + \beta Y + \gamma Z)\right)\right)^{\otimes 5} \\ &= ((1-\varepsilon)|T_0\rangle\langle T_0| + \varepsilon|T_1\rangle\langle T_1|)^{\otimes 5} \\ &= \rho_1(\varepsilon) \end{aligned}$$

and

$$D^{\otimes 5}\left(p\rho_0(\alpha,\beta,\gamma)+(1-p)\rho_0(\alpha',\beta',\gamma')\right)^{\otimes 5}=D^{\otimes 5}\left(\frac{1}{2}(I+\alpha''X+\beta''Y+\gamma''Z)\right)^{\otimes 5}=\rho_1(\varepsilon'')$$

for some $\varepsilon''\in[0,1]$.

Next, for any $\varepsilon\in[0,1]$,

$$\begin{aligned}\Phi_2(\rho_1(\varepsilon)) =&(I\otimes\langle 0000|)\,U_{\text{dec}}\,\mathsf{E}_{\text{acc}}\,\left((1-\varepsilon)\,|T_0\rangle\,\langle T_0|+\varepsilon\,|T_1\rangle\,\langle T_1|\right)\,\mathsf{E}_{\text{acc}}\,U_{\text{dec}}^{\dagger}\,(I\otimes|0000\rangle)\\ =&p_{\text{acc}}(\varepsilon)(1-\varepsilon_{\text{out}}(\varepsilon))\,|T_0\rangle\,\langle T_0|+p_{\text{acc}}(\varepsilon)\varepsilon_{\text{out}}(\varepsilon)\,|T_1\rangle\,\langle T_1|\\ =&\rho_2(p_{\text{acc}}(\varepsilon),\varepsilon_{\text{out}}(\varepsilon))\end{aligned}$$

- For each effect $E_1\in\mathsf{Eff}_1$, $\Phi_1^*(E_1)\in\mathsf{Eff}_0$ and, for each effect $E_2\in\mathsf{Eff}_2$, $\Phi_2^*(E_2)\in\mathsf{Eff}_1$.
  - For any $E_1\in\mathsf{Eff}_1$,
    * if $E_1\in\mathscr{G}_1$, $\Phi_1^*(E_1)=\frac{1}{3^5}\sum_{\alpha\in\{-1,0,1\}^5}\left(T^{\otimes\alpha}\right)^{\dagger}E_1T^{\otimes\alpha}\in\mathscr{G}_0$
    * if E_1 is the product of finite set of commuting effects in $\mathscr{G}_1$, then we have $E_1\in P_1$, and by definition of G_0, $\Phi_1^*(E_1)\in G_0$.
    * if E_1 is a convex sum of effects in Eff_1, then since Φ_1^* is linear and Eff_0 is closed under convex sum, and assuming that each component of the sum is transformed into an effect in Eff_0, it follows that $\Phi_1^*(E_1)$ is in Eff_0.
  - For any $E_2\in\mathsf{Eff}_2$,
    * if $E_2\in\mathscr{G}_2$,

$$\begin{aligned}\Phi_2^*(E_2)&=\mathsf{E}_{\text{acc}}\,U_{\text{dec}}^{\dagger}\,(I\otimes|0000\rangle)\,E_2\,(I\otimes\langle 0000|)\,U_{\text{dec}}\,\mathsf{E}_{\text{acc}}\\ &=\mathsf{E}_{\text{acc}}\,U_{\text{dec}}^{\dagger}\,(E_2\otimes|0000\rangle\,\langle 0000|)\,U_{\text{dec}}\,\mathsf{E}_{\text{acc}}\\ &=\begin{cases}\mathsf{E}_{\text{acc}}\,U_{\text{dec}}^{\dagger}\,((|T_0\rangle\,\langle T_0|)\otimes|0000\rangle\,\langle 0000|)\,U_{\text{dec}}\,\mathsf{E}_{\text{acc}}, & E_2=|T_0\rangle\,\langle T_0|\\ \mathsf{E}_{\text{acc}}\,U_{\text{dec}}^{\dagger}\,((|T_1\rangle\,\langle T_1|)\otimes|0000\rangle\,\langle 0000|)\,U_{\text{dec}}\,\mathsf{E}_{\text{acc}}, & E_2=|T_1\rangle\,\langle T_1|\end{cases}\\ &=\begin{cases}\mathsf{E}_{\text{acc}}\,\left|T_1^L\right\rangle\left\langle T_1^L\right|\,\mathsf{E}_{\text{acc}}, & E_2=|T_0\rangle\,\langle T_0|\\ \mathsf{E}_{\text{acc}}\,\left|T_0^L\right\rangle\left\langle T_0^L\right|\,\mathsf{E}_{\text{acc}}, & E_2=|T_1\rangle\,\langle T_1|\end{cases}\\ &=\begin{cases}\left|T_1^L\right\rangle\left\langle T_1^L\right|, & E_2=|T_0\rangle\,\langle T_0|\\ \left|T_0^L\right\rangle\left\langle T_0^L\right| & E_2=|T_1\rangle\,\langle T_1|\end{cases}\end{aligned}$$

      where $\{\left|T_0^L\right\rangle\left\langle T_0^L\right|,\ \left|T_1^L\right\rangle\left\langle T_1^L\right|\}\subseteq\mathsf{Eff}_1$.
    * if E_2 is the product of a finite commuting set of effects in $\mathscr{G}_2$, then it follows that $E_2\in\mathscr{G}_2$, $E_2=0$, or $E_2=I$. Since the first case is covered in the previous case, it suffices to show that $\Phi_2^*(0)\in\mathsf{Eff}_1$ and $\phi_2^*(I)\in\mathsf{Eff}_1$. Since $\Phi_2^*(0)=0$, it follows that $\Phi_2^*(0)\in\mathscr{P}_1\subseteq\mathsf{Eff}_1$. Also, since $\Phi_2^*(I)=\mathsf{E}_{\text{acc}}=\prod_{S\in\mathscr{S}}P_{S,+1}\in\mathscr{P}_1$ it follows that $\Phi_2^*(I)\in\mathsf{Eff}_1$.
    * if E_2 is a convex sum of effects in Eff_2, then since Φ_2^* is linear and Eff_1 is closed under convex sum, and assuming that each component of the sum is transformed into an effect in Eff_1, it follows that $\Phi_2^*(E_2)$ is in Eff_1.

Next, we show that R_0, R_1, and R_2 are valid polynomial spectral signal-state relations, and that

(f_1,Φ_1) and (f_2,Φ_2) are morphisms between the polynomial spectral signal-state relations. Recall that

$$R_0 = \left(M_0 = \{(\alpha,\beta,\gamma)\in\mathbb{R}^3 \mid \alpha^2+\beta^2+\gamma^2 \le 1\},\ X_0,\ \rho_0 : (\alpha,\beta,\gamma) \mapsto \left(\frac{1}{2}(I+\alpha X+\beta Y+\gamma Z)\right)^{\otimes 5}\right)$$
$$R_1 = \left(M_1 = [0,1],\ X_1,\ \rho_1 : \varepsilon \mapsto ((1-\varepsilon)\left|T_0\right\rangle\left\langle T_0\right| + \varepsilon\left|T_1\right\rangle\left\langle T_1\right|)^{\otimes 5}\right)$$
$$R_2 = \left(M_2 = [0,1]^2,\ X_2,\ \rho_2 : (\mu,\varepsilon) \mapsto \mu(1-\varepsilon)\left|T_0\right\rangle\left\langle T_0\right| + \mu\varepsilon\left|T_1\right\rangle\left\langle T_1\right|\right)$$

First, to show that R_0, R_1, and R_2 are valid polynomial spectral signal-state relations, we need to show that the functions $e_{E_i} : m \mapsto \mathrm{Tr}(E_i\rho_i(m))$ are polynomials, for all $m \in M_i$ and $E_i \in \mathsf{Eff}_i$, for $i = 0,1,2$. Note that the function e_{E_i} is linear on the effect E_i. Therefore, it suffices to show that e_{E_i} is polynomial for each effect $E_i \in \mathscr{P}_i$. However, since each entry of the matrix $\rho_i(m)$ is polynomial in the corresponding variables of the coordinates of m and each $e_{E_i} \in \mathscr{P}_i$ is constant matrix, it follows that the trace $e_{E_i} = \mathrm{Tr}(E_i\rho_i(m))$ is polynomial.

Next, we need to show that the morphisms (f_1,Φ_1) and (f_2,Φ_2) for

$$f_1 : (\alpha,\beta,\gamma) \mapsto \mathrm{Tr}\left(\left|T_1\right\rangle\left\langle T_1\right| D\left(\frac{1}{2}(I+\alpha X+\beta Y+\gamma Z)\right)\right)$$

and

$$f_2 : \varepsilon \mapsto (p_{\mathrm{acc}}(\varepsilon), \varepsilon_{\mathrm{out}}(\varepsilon))$$

satisfy the following conditions:

- for all $E_1 \in \mathsf{Eff}_1$ and all $(\alpha,\beta,\gamma) \in M_0$,

$$\begin{aligned}
\mathrm{Tr}(\Phi_1^*(E_1)\rho_0(\alpha,\beta,\gamma)) &= \frac{1}{3^5}\sum_{\alpha\in\{-1,0,1\}^5} \mathrm{Tr}\left(\left(T^{\otimes\alpha}\right)^\dagger E_1 T^{\otimes\alpha}\rho_0(\alpha,\beta,\gamma)\right)\\
&= \frac{1}{3^5}\sum_{\alpha\in\{-1,0,1\}^5} \mathrm{Tr}\left(E_1 T^{\otimes\alpha}\rho_0(\alpha,\beta,\gamma)\left(T^{\otimes\alpha}\right)^\dagger\right)\\
&= \mathrm{Tr}\left(E_1 \frac{1}{3^5}\sum_{\alpha\in\{-1,0,1\}^5} T^{\otimes\alpha}\left(\frac{1}{2}(I+\alpha X+\beta Y+\gamma Z)\right)^{\otimes 5}\left(T^{\otimes\alpha}\right)^\dagger\right)\\
&= \mathrm{Tr}\left(E_1\left(D\left(\frac{1}{2}(I+\alpha X+\beta Y+\gamma Z)\right)\right)^{\otimes 5}\right)\\
&= \mathrm{Tr}\left(E_1\left((1-f_1(\alpha,\beta,\gamma))\left|T_0\right\rangle\left\langle T_0\right| + f_1(\alpha,\beta,\gamma)\left|T_1\right\rangle\left\langle T_1\right|\right)^{\otimes 5}\right)\\
&= \mathrm{Tr}(E_1\rho_1(f_1(\alpha,\beta,\gamma)))
\end{aligned}$$

- for all $E_2 \in \mathsf{Eff}_2 = \{\left|T_0\right\rangle\left\langle T_0\right|,\ \left|T_1\right\rangle\left\langle T_1\right|\}$ and all $m_1 = \varepsilon \in M_1$,

- if $E_2 = |T_0\rangle\langle T_0| \in \mathscr{G}_2$:

$$\begin{aligned}
\mathrm{Tr}(\Phi_2^*(E_2)\rho_1(m_1)) &= \mathrm{Tr}(\Phi_2^*(|T_0\rangle\langle T_0|)\rho_1(\varepsilon)) \\
&= \mathrm{Tr}(|T_1^L\rangle\langle T_1^L|((1-\varepsilon)|T_0\rangle\langle T_0| + \varepsilon|T_1\rangle\langle T_1|)^{\otimes 5}) \\
&= \mathrm{Tr}\left(\mathsf{E}_{\mathrm{acc}}|T_1^L\rangle\langle T_1^L|\mathsf{E}_{\mathrm{acc}}\left(\sum_{x\in\{0,1\}^5}(1-\varepsilon)^{5-|x|}\varepsilon^{|x|}|T_x\rangle\langle T_x|\right)\right) \\
&= \langle T_1^L|\left(\sum_{x\in\{0,1\}^5}(1-\varepsilon)^{5-|x|}\varepsilon^{|x|}\mathsf{E}_{\mathrm{acc}}|T_x\rangle\langle T_x|\mathsf{E}_{\mathrm{acc}}\right)|T_1^L\rangle \\
&= \frac{(1-\varepsilon)^5}{6} + \frac{5(1-\varepsilon)^2\varepsilon^3}{6} \\
&= p_{\mathrm{acc}}(\varepsilon)(1-\varepsilon_{\mathrm{out}}(\varepsilon)) \\
&= \mathrm{Tr}(p_{\mathrm{acc}}(\varepsilon)(1-\varepsilon_{\mathrm{out}}(\varepsilon))|T_0\rangle\langle T_0|) \\
&= \mathrm{Tr}(|T_0\rangle\langle T_0|(p_{\mathrm{acc}}(\varepsilon)(1-\varepsilon_{\mathrm{out}}(\varepsilon))|T_0\rangle\langle T_0| + p_{\mathrm{acc}}(\varepsilon)\varepsilon_{\mathrm{out}}(\varepsilon)|T_1\rangle\langle T_1|)) \\
&= \mathrm{Tr}(|T_0\rangle\langle T_0|\rho_2(p_{\mathrm{acc}}(\varepsilon),\varepsilon_{\mathrm{out}}(\varepsilon))) \\
&= \mathrm{Tr}(E_2\rho_2(f_2(m_1)))
\end{aligned}$$

- if $E_2 = |T_1\rangle\langle T_1| \in \mathscr{G}_2$:

$$\begin{aligned}
\mathrm{Tr}(\Phi_2^*(E_2)\rho_1(m_1)) &= \mathrm{Tr}(\Phi_2^*(|T_1\rangle\langle T_1|)\rho_1(\varepsilon)) \\
&= \mathrm{Tr}(|T_0^L\rangle\langle T_0^L|((1-\varepsilon)|T_0\rangle\langle T_0| + \varepsilon|T_1\rangle\langle T_1|)^{\otimes 5}) \\
&= \mathrm{Tr}\left(\mathsf{E}_{\mathrm{acc}}|T_0^L\rangle\langle T_0^L|\mathsf{E}_{\mathrm{acc}}\left(\sum_{x\in\{0,1\}^5}(1-\varepsilon)^{5-|x|}\varepsilon^{|x|}|T_x\rangle\langle T_x|\right)\right) \\
&= \langle T_0^L|\left(\sum_{x\in\{0,1\}^5}(1-\varepsilon)^{5-|x|}\varepsilon^{|x|}\mathsf{E}_{\mathrm{acc}}|T_x\rangle\langle T_x|\mathsf{E}_{\mathrm{acc}}\right)|T_0^L\rangle \\
&= \frac{5(1-\varepsilon)^3\varepsilon^2}{6} + \frac{\varepsilon^5}{6} \\
&= p_{\mathrm{acc}}(\varepsilon)\varepsilon_{\mathrm{out}}(\varepsilon) \\
&= \mathrm{Tr}(p_{\mathrm{acc}}(\varepsilon)\varepsilon_{\mathrm{out}}(\varepsilon)|T_1\rangle\langle T_1|) \\
&= \mathrm{Tr}(|T_1\rangle\langle T_1|(p_{\mathrm{acc}}(\varepsilon)(1-\varepsilon_{\mathrm{out}}(\varepsilon))|T_0\rangle\langle T_0| + p_{\mathrm{acc}}(\varepsilon)\varepsilon_{\mathrm{out}}(\varepsilon)|T_1\rangle\langle T_1|)) \\
&= \mathrm{Tr}(|T_1\rangle\langle T_1|\rho_2(p_{\mathrm{acc}}(\varepsilon),\varepsilon_{\mathrm{out}}(\varepsilon))) \\
&= \mathrm{Tr}(E_2\rho_2(f_2(m_1)))
\end{aligned}$$

- if E_2 is the product of finite set of commuting effects in $\mathscr{G}_2$, then it follows that E_2 can be 0, I, or in the generating effects $\mathscr{G}_2$. The case where E_2 is in $\mathscr{G}_2$ is covered by the previous cases.

$$\mathrm{Tr}(\Phi_2^*(0)\rho_1(\varepsilon)) = 0 = \mathrm{Tr}(0\rho_2(f_2(\varepsilon)))$$

and

$$\mathrm{Tr}(\Phi_2^*(I)\rho_1(\varepsilon)) = \mathrm{Tr}((|T_0^L\rangle\langle T_0^L| + |T_1^L\rangle\langle T_1^L|)\rho_1(\varepsilon)) = p_{\mathrm{acc}}(\varepsilon) = \mathrm{Tr}(I\rho_2(f_2(\varepsilon)))$$

- $E_2 = pE_{2,1} + (1-p)E_{2,2}$ for $p \in [0,1]$ and $E_{2,1}, E_{2,2} \in \mathsf{Eff}_2$:

$$\begin{aligned}
\mathrm{Tr}(\Phi_2^*(E_2)\rho_1(m_1)) &= \mathrm{Tr}(\Phi_2^*(pE_{2,1} + (1-p)E_{2,2})\rho_1(m_1)) \\
&= \mathrm{Tr}(p\Phi_2^*(E_{2,1})\rho_1(m_1) + (1-p)\Phi_2^*(E_{2,2})\rho_1(m_1)) \\
&= p\mathrm{Tr}(\Phi_2^*(E_{2,1})\rho_1(m_1)) + (1-p)\mathrm{Tr}(\Phi_2^*(E_{2,2})\rho_1(m_1)) \\
&= p\mathrm{Tr}(E_{2,1}\rho_2(f_2(m_1))) + (1-p)\mathrm{Tr}(E_{2,2}\rho_2(f_2(m_1))) \\
&= \mathrm{Tr}((pE_{2,1} + (1-p)E_{2,2})\rho_2(f_2(m_1))) \\
&= \mathrm{Tr}(E_2\rho_2(f_2(m_1)))
\end{aligned}$$

□