Structure and geometry complete completeness

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***Abstract*—We propose a structural perspective on completeness for equational theories of quantum computing, observing that a large class of equations arise from ambient computational structure, notably that of rig categories. Once this structure is accounted for, the remaining equations are few and admit a clear geometric interpretation. We make this perspective concrete by giving a constructive and generic description of the equational theory for unitary matrices, explicating generic normal forms for unitaries, and use this to derive a recipe to obtain complete equational theories for unitaries with any generator, as long as we have completeness on 2-dimensional systems and enough structure.**

I. Introduction

Graphical languages have become a standard tool for representing quantum processes, ranging from quantum circuits [1] to more expressive calculi such as the flexsymmetric [2] ZX- [3], ZW- [4], and ZH-calculus [5], all rooted in categorical quantum mechanics [6]. While these descriptions appear simple, their underlying mathematics is not. This is reflected in the long timeline of completeness results: the first complete equational theory for the ZX-calculus only appeared in 2018 [7], and a complete theory for (qubit) quantum circuits followed even later [8].

These developments suggest that completeness is not merely a matter of adding enough equations, but rather of identifying the right underlying principles. Indeed, existing completeness results often arise by transporting structure between different formalisms. For instance, the completeness of quantum circuits was obtained via a translation from the LOPP-calculus, a *direct-sum*-based graphical language closely related to graphical linear algebra [9], designed to model unitary linear optical circuits. While technically effective, such translations obscure the conceptual origin of the equations involved.

A closer look reveals that many equations do not need to be postulated at all: they arise automatically from ambient computational structures. Quantum circuits, for example, form a prop [11] whose monoidal structure captures the parallel composition of qubits. As a consequence, a large class of equations, such as bifunctoriality (see (1) in Figure 1), which holds by construction, rather than by axiom. In contrast, in direct-sum-based languages such as the LOPP-calculus, the analogous equation (2) is not immediate, yet must still be derivable due to completeness. Similarly, translating the structural equations related to the direct sum into quantum circuits is not trivial, but it turns out to be entirely linked to the notion of control [12], and can be summarised in eight simple equations.

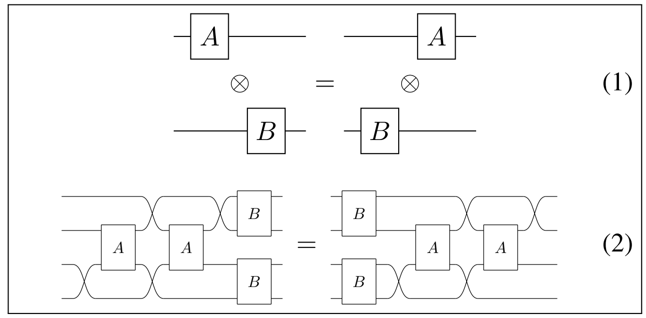


Fig. 1. Structural rule that, along with (G-E2), can derive[4] (G-E3) on at most four wires. A and B would be all 2-dimensional generator chosen to be in the prop. The second rule is the direct-sum analogy of the bifunctiorilaity of the Kronecker product, *i.e.* the first equation (1), which is native in a prop for quantum circuits.

This contrast points to a more general phenomenon. Across different graphical languages and formalisms, a substantial portion of the equational theory is determined by purely structural considerations. A ubiquitous computational structure is that of rig categories [13, 14], which appear in everything from models of the lambda calculus [14, 15] to reversible computing [16–18] and quantum programming [19–24].

What, then, remains once this structure is accounted for? Our key observation is that the remaining equations are few, and that they admit a clear geometric interpretation. In the case of quantum computing, they arise from the geometry of the Bloch sphere alone. This leads us to the guiding principle:

completeness = structure + geometry.

Rather than viewing completeness as a combinatorial or syntactic property, we propose to understand it as the consequence of two interacting ingredients: a rigid structural backbone, and a geometric layer capturing the specific nature of the systems under consideration.

a) State of the art: The first explicit graphical equational theory only complete for the unitary fragment has been introduced with the LOPP-calculus [25]. In this framework, the two linear optical components, i.e. phase shifter and beam splitter, are semantically the phase and 2×2-matrix R_X rotations[3], while the parallel composition is the direct sum. Following that result, a back-and-forth translation gave a complete equational theory with similar generators for quantum circuits [8], i.e. unitary matrices with the Kronecker product for the parallel composition. Since then, simplified and minimal equational

[3]Recall the definition: $R_X(\theta) = \begin{bmatrix} \cos(\theta/2) & -i\sin(\theta/2) \\ -i\sin(\theta/2) & \cos(\theta/2) \end{bmatrix}$

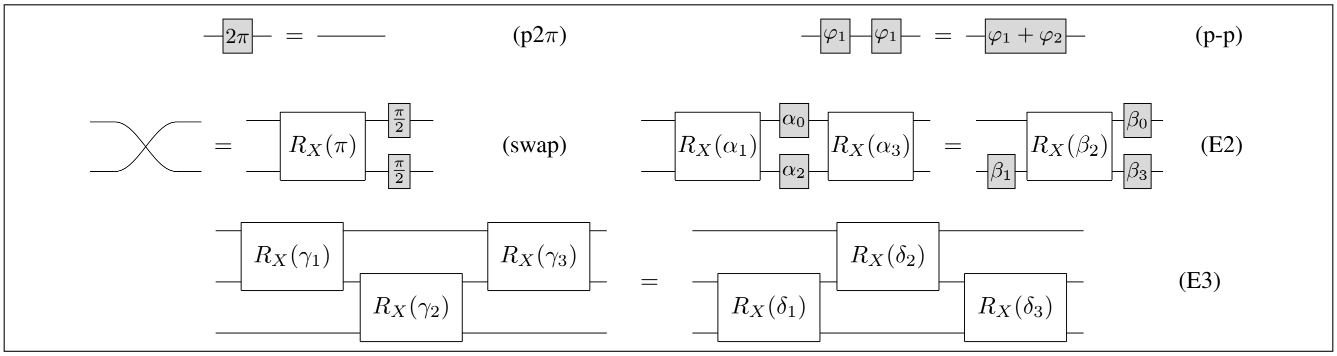


Fig. 2. Minimal equational theory for the direct-sum unitary fragment [10], with phases and R_X rotations as generators.

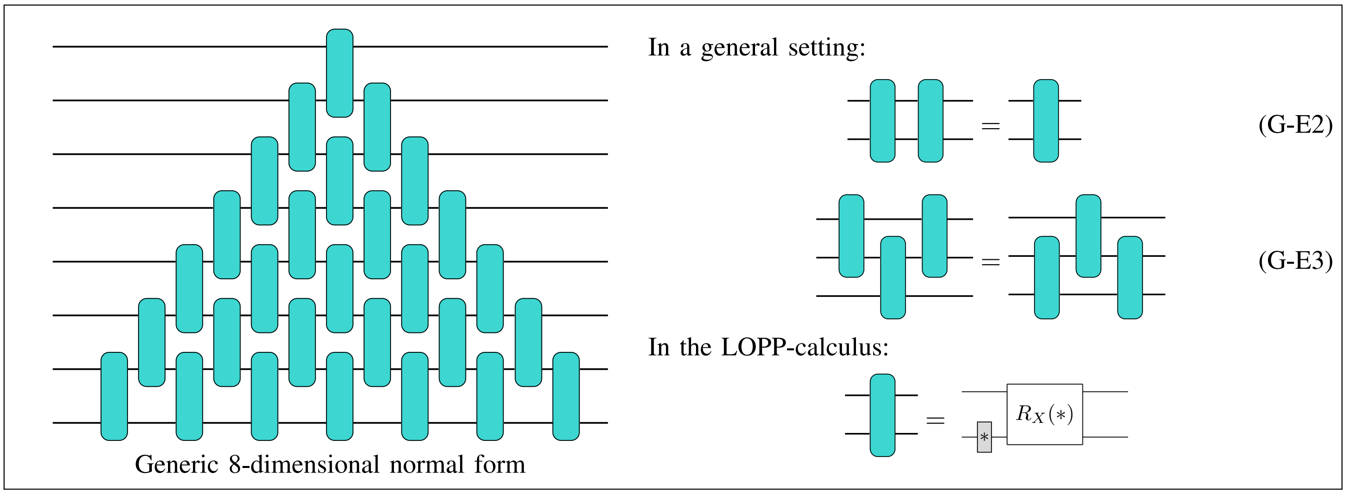


Fig. 3. Each 2-mode block is a unique normal form, universal for 2×2 unitary matrices[3]. Every unitary can be uniquely decomposed as a triangle composed of these 2-mode block[3]. The normal form is obtained[2]with the oriented versions (from left to right) of (G-E2) and (G-E3). In the LOPP-calculus, this block would consist of one phase and one R_X rotation, cf Definition 9 of [10].

theories have been provided, both for the direct sum [10], that is depicted in Figure 2, and Kronecker product [26].

While the theories for direct-sum circuits [10] and (qubit) quantum circuits [26] share many similarities — they have similar generators and a family of 2-dimensional Euler equations[4] — there is one key difference: (E3), depicted in Figure 2, is missing in the minimal (qubit) quantum circuits theories [26]. This is a surprising result, as (E3) has proven to be necessary with a direct-sum composition [10]. More broadly, there is also the question of similar theories that involve different generators, e.g. Hadamard [27] instead of R_X gates.

b) Contributions: In this work, we study a general framework for characterising equational theories of unitaries while exhibiting when (E3) is necessary. More specifically, we provide:

- A constructive and generic description of the equational theory for the direct-sum composition, explaining generic normal forms for unitaries (Section II).
- The link between (E3) and the bifunctoriality of the Kronecker product, explaining why it is not necessary in (qubit) quantum circuits theories as this property is native for such tensored structures (Section III).
- A recipe to obtain complete equational theories for unitaries with any generator, as long as we have completeness on 2-dimensional wires and enough structure, *e.g.* a rig category (see Theorem 4).

[4]An infinite number of instances have been proven to be necessary [26].

II. Unitaries with direct-sum composition: Generic normal forms

The proof of completeness for direct-sum circuits [25] is constructive and relies on a deterministic rewriting system, converging to a triangular-shaped normal form. This technique may appear specific to phases and R_X rotations, but we can generalise. As long as we have a 2-dimensional unique normal form, then we can prove a unique normal form for arbitrary dimensions with the same triangular shape. This shape and the rewriting rules are depicted in Figure 3. Note that the equation (G-E2) can be both seen as an equation reducing the number of blocks, and as a generalisation of (E2). Similarly, the equation (G-E3) helps rearrange the blocks in a 3-dimensional normal form, and is a generalisation of (E3).

[3]Up to phases on the right.

[4]Up to propagation of phases to the right.

Lemma 1. *A minimal complete equational theory for direct-sum unitary circuits consists of:*

- *Completeness on 1 wire: e.g. (p2π) and (p-p).*
- *Completeness on 2 wires:*
  - *the 2d normal form for each generator, e.g. (swap),*
  - *the 2d normal form of two sequential 2d normal forms, i.e. (G-E2).*
- *Triangular rearrangement of three 2d normal forms, i.e. (G-E3).*

The equations on 1 and 2 wires have clear geometric interpretations. Since Z-axis rotations are made up of phases alone, axioms (p2π) and (p-p) state that rotations are periodic and compose by adding angles. Likewise, the (swap) rule posits that the symmetry is an X-rotation (up to a global phase), while (G-E2) is the familiar Euler decomposition.

We say that a prop P is *universal* if it admits a full monoidal functor into unitaries. This universality is central in this work, because the normal forms rely on having access to all phases.

The observations above allow for a deeper theorem, which does not include any mention of the generators.

Theorem 2. *Let P the free prop generated with arbitrary 2-dimensional generators and phases. If P is universal for unitary matrices, then an equational theory on P, complete for up to 3-dimensional unitaries, is complete for all unitaries.*

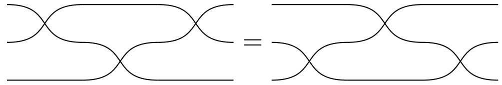
This theorem generalises the completeness result for the LOPP-calculus. In fact, only phases are strictly necessary, while completeness for 2-dimensional unitaries can be achieve with (almost) any set of 2-wire gates, and it is sufficient that any 2-dimensional unitary has an Euler-like decomposition. Because Euler decompositions depend only on computations on the Bloch sphere, we refer to this part of the theory as the *geometry of the qubit*.

III. About the structural derivation of (E3)

One of our main contributions is that the equation (E3), or more generally (G-E3), is in fact structural in complete theories for (qubit) unitary quantum circuits [26]. It turns out that (G-E3) can be derived with the use of (G-E2) and a new rule (2), depicted in Figure 1. This rule is analogous to the Kronecker bifunctoriality equation (1).

Lemma 3. *On circuits with at least four wires, we can derive[4] (G-E3) from (G-E2) and (2). Reciprocally, we can derive[4] (G-E2) and (2) from (G-E3).*

This echoes equations for reversible Boolean circuits, where the 3-dimensional equation is known as the Yang-Baxter equation, as pictured below.


(3)

This equation is structural for the direct sum, and therefore structural for a rig structure. Lemma 3 allows us to push this story further and assert that the 3-dimensional equation (G-E3) is structural when the geometry of the qubit is known. We can therefore view (G-E3) as a unitary generalisation of the Yang-Baxter equation.

While completeness for classical reversible circuit is strictly structural, we observe that completeness for unitaries requires only structure and geometry of the qubit. Structure here refers to the equations hidden under the hood of the direct sum, to which we add the bifunctoriality of the Kronecker product (2). Moreover, the geometry of the qubit is entirely captured by an equation of the form (G-E2), allowing for an Euler-like decomposition for all 2-dimensional unitaries.

Following Lemma 1 and 3, we can prove the following.

Theorem 4. *Let P the free prop generated with arbitrary 2-dimensional generators and phases. If P is universal for unitary matrices, then an equational theory on P, complete for up to 2-dimensional unitaries and which also contains (2), is complete for unitaries of 4 dimensions and above.*

IV. Conclusion

In this work in progress, we focus on direct-sum unitary circuits and their completeness equational theories. We generalise previous results in this area, first showing that completeness on up to 3-dimensional unitaries is enough to derive completeness for all unitaries. We then further observe that 3-dimensional equations are structural for rig categories, and particularly derivable with the bifunctoriality of the Kronecker product. This leads us to the guiding principle for unitary completeness:

completeness = structure + geometry.

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