Operator Spaces, Linear Logic and the Heisenberg-Schrödinger Duality of Quantum Theory

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***Abstract*—We show that the category OS of operator spaces, with complete contractions as morphisms, is locally countably presentable and a model of Intuitionistic Linear Logic in the sense of Lafont. We then describe a model of Classical Linear Logic, based on OS, whose duality is compatible with the Heisenberg-Schrödinger duality of quantum theory. We explore connections between the Haagerup tensor product on OS and BV-logic and we also show that OS provides a good setting for studying pure state and mixed state quantum information, the interaction between the two, and even higher-order quantum maps such as the quantum switch. This work was published in LICS'25 [1]. A preprint with many omitted proofs is available [2] and we also have a detailed companion paper [3] that provides a proof of the local presentability of OS and proofs that establish other important categorical properties of OS.**

I. Introduction

The Heisenberg-Schrödinger duality establishes a duality between two pictures of quantum theory. In the Schrödinger picture, it is often said that one modifies the state of the quantum system while keeping the observable invariant. In the Heisenberg picture, it is the other way round, i.e. the observable is modified while keeping the state invariant. There is a rich mathematical theory, based on functional analysis [4], noncommutative geometry [5], and operator algebras (e.g. von Neumann algebras) [6], [7], [8], [9], that can be used to describe this duality and that can make precise what we mean by "state" and "observable".

What is missing from the picture (figuratively) is whether this duality can be understood using Linear Logic [10]. Indeed, Linear Logic (LL) has been influential in our understanding of dualities, because it provides us with a rich logical setting, equipped with numerous logical connectives, that allow us to study dualities. This is reflected quite nicely via Polarised Linear Logic [11] which makes it clear how certain LL formulas may be assigned positive or negative logical polarities (see also [12] for more information). Furthermore, the semantics of linear logic has been extensively studied and the relevant categorical models are well-known [13], [14], [15] which gives us another way to reason about the relevant dualities.

Our paper is concerned with the following question:

> Can we use linear logic semantics to study the Heisenberg-Schrödinger duality (in infinite dimensions)?

We provide strong evidence in support of an affirmative answer. In particular, we construct a model of LL whose duality is compatible with the Heisenberg-Schrödinger duality and where system descriptions in the Schrödinger picture correspond to formulas with a positive logical polarity, whereas those in the Heisenberg picture correspond to formulas with a negative logical polarity. In the process of constructing this model, we prove mathematical results in the intersection of noncommutative geometry and category theory, via the theory of operator spaces [16], [17], [18].

II. Banach spaces and the Heisenberg-Schrödinger duality

Banach spaces, i.e., complete normed vector spaces (which we assume to be complex valued), form the foundational structures of quantum physics. For example, a quantum system is typically described by a Hilbert space H, which is an instance of a Banach space. Two other examples of Banach spaces are:

- $B(H)$, the algebra of all bounded operators on H. Since this space is spanned by the selfadjoint operators, representing the system's observables, it is the natural setting for the *Heisenberg picture* of quantum physics, where the system's time evolution is described in terms of observables.
- $T(H)$, the space of trace-class operators on H. Since it is spanned by the density operators, representing the (mixed) states of the system, this space is fundamental to the *Schrödinger picture* of quantum physics, in which time evolution is described in terms of states.

Mathematically, the duality between both pictures follows from the isometric isomorphism between $B(H)$ and the Banach space dual $T(H)^*$ of $T(H)$. In the language of von Neumann algebras: $T(H)$ is isometrically isomorphic to the *predual* $B(H)_*$ of $B(H)$. For simplicity of the presentation, we view this isomorphism as an equality $T(H)^* = B(H)$. It follows that the map $\psi \mapsto \psi^*$ is an isometric isomorphism between bounded maps $T(H_1) \to T(H_2)$ and normal (=weak*-continuous) linear maps $B(H_2) \to B(H_1)$. In fact, operator spaces refine this form of the Heisenberg-Schrödinger duality in several ways and make it more appealing from a categorical and logical point of view.

III. Operator space models of linear logic

The category $\mathbf{Ban}$ of Banach spaces and linear contractions is symmetric monoidal closed with respect to the *projective* tensor product of Banach spaces $\overset{p}{\otimes}$. The internal hom of Banach spaces X and Y is the Banach space $B(X,Y)$ of bounded operators $X \to Y$. Furthermore, $\mathbf{Ban}$ is a model of Intuitionistic Linear Logic, because it has *Lafont exponentials*. Writing $\mathbf{CoComon}$ for the category of cocommutative comonoids internal to $(\mathbf{Ban}, \overset{p}{\otimes}, \mathbb{C})$, this means that the forgetful functor $\mathbf{CoComon} \to \mathbf{Ban}$ has a right adjoint. The induced comonad models the exponential modality of linear logic. The existence of the adjunction follows because $\mathbf{Ban}$ is *locally presentable* and symmetric monoidal closed using results from [19]. Recall that a category is locally presentable if it is locally small, cocomplete, and there is a cardinal λ and a small set of λ-compact objects that generate the category via λ-filtered colimits [20], [21].

However, $\overset{p}{\otimes}$ does not correctly describe the composition of quantum systems, since the Hilbert space tensor product $H \otimes K$ is not equal to $H \overset{p}{\otimes} K$ (in general in infinite-dimensions). Furthermore, in general, we have $T(H \otimes K) \ncong T(H) \overset{p}{\otimes} T(K)$ and $B(H \otimes K) \ncong B(H) \overset{p}{\otimes} B(K)$. The reason for this mismatch with quantum theory is that Banach spaces are essentially classical mathematical structures and one needs to use more elaborate tensor products coming from Noncommutative Geometry [5]. The program of Noncommutative Geometry asserts that quantum phenomena should be described by structures of (noncommuting) operators on Hilbert spaces. Usually, quantum phenomena have classical counterparts, and can be described by noncommutative generalisations of the structures describing these counterparts. *Operator Spaces* are the noncommutative generalisation of Banach spaces and they are of fundamental importance for the results that we establish.

Definition III.1. [22], [18] A (concrete) *operator space* is a norm-closed subspace $X \subseteq B(H)$ for some Hilbert space H. If $Y \subseteq B(K)$ is another operator space, a linear map $u : X \to Y$ is called *completely bounded* if there is some Hilbert space $\hat{H}$, a unital $*$-homomorphism $\pi : B(H) \to B(\hat{H})$ and bounded linear maps $v_1, v_2 : K \to \hat{H}$ such that $u(x) = v_2^* \pi(x) v_1$ for each $x \in X$. If v_1 and v_2 can be chosen to be contractions, then u is called a *complete contraction*. If $v_1 = v_2$, then u is called *completely positive*. We write $\mathbf{OS}$ for the category of (possibly infinite-dimensional) operator spaces with complete contractions as morphisms.

Any Banach space X is isometrically embeddable into $B(H)$ for some Hilbert space H, and can hence be regarded as an operator space. However, one has to keep track of the embedding as various choices are possible. Any such choice is called an *operator space structure (OSS)* on X. Equivalently, one can define operator spaces as Banach spaces equipped with an OSS. Both $T(H)$ and $B(H)$ have a canonical OSS, and can be regarded as operator spaces. The *operator space projective tensor product* $\hat{\otimes}$ on $\mathbf{OS}$ is the operator space counterpart of $\overset{p}{\otimes}$ on $\mathbf{Ban}$. We prove that $\mathbf{OS}$ is symmetric monoidal closed with respect to $\hat{\otimes}$ with internal hom $CB(X,Y)$, the operator space of completely bounded maps $X \to Y$. Moreover, $\hat{\otimes}$ correctly describes the composition of quantum systems in the following sense: any Hilbert space H becomes an operator space H_c when equipped with a canonical OSS that captures the inner product of H and then $(H \otimes K)_c \cong H_c \hat{\otimes} K_c$, and $T(H \otimes K) \cong T(H) \hat{\otimes} T(K)$. One of our main results is given next.

Theorem III.2. *The category* $\mathbf{OS}$ *is: (1) locally presentable; (2) symmetric monoidal closed; (3) a model of Intuitionistic Linear Logic.*

A model of full linear logic can be obtained by using the *Chu construction.*

Definition III.3. We write $\mathbf{Q}$ for $\mathbf{Chu}(\mathbf{OS}, \mathbb{C})$, the Chu category of $\mathbf{OS}$ with dualising object given by $\mathbb{C}$. That is, objects are triples (X, Y, d) where X and Y are operator spaces and $d \colon X \hat{\otimes} Y \to \mathbb{C}$ is a complete contraction; a morphism is a pair $(f,g) \colon (X_1, Y_1, d_1) \to (X_2, Y_2, d_2)$ where $f \colon X_1 \to X_2$ and $g \colon Y_2 \to Y_1$ are complete contractions such that $d_1 \circ (\mathrm{id}_{X_1} \hat{\otimes} g) = d_2 \circ (f \hat{\otimes} \mathrm{id}_{Y_2})$.

The next theorem follows by using results of Barr [23].

Theorem III.4. *The category* $\mathbf{Q}$ *is complete, cocomplete, $*$-autonomous, has a Lafont exponential and it is therefore a model of full linear logic. The dual in* $\mathbf{Q}$ *is defined as* $(X, Y, d)^\perp \overset{def}{=} (Y, X, d \circ \sigma)$ *on objects, where* $\sigma \colon Y \hat{\otimes} X \xrightarrow{\cong} X \hat{\otimes} Y$ *is the symmetry. On morphisms, it is simply* $(f,g)^\perp \overset{def}{=} (g, f)$.

This theorem allows us to view a version of the Heisenberg-Schrödinger duality in a more categorical and logical way. We discuss this in more detail in the next section.

IV. Operator Spaces and the Heisenberg-Schrödinger duality

The duality between the Heisenberg and the Schrödinger pictures can be refined in the operator space framework. For a bounded map $\psi \colon T(H_1) \to T(H_2)$ and its adjoint (i.e. dual map) $\psi^* \colon B(H_2) \to B(H_1)$, we have

1) ψ is (completely-)positive iff ψ^* is (completely)-positive;
2) ψ is trace-preserving iff ψ^* is unital;
3) ψ is a (complete) contraction iff ψ^* is a (complete) contraction.

In fact, if ψ is trace-preserving, then ψ is completely positive iff ψ is a complete contraction. Dually, if ψ^* is unital, then ψ^* is completely positive iff ψ^* is a complete contraction. A pair $(\varphi, \psi) \colon (T(H_1), B(H_1), \mathrm{tr}) \to (T(H_2), B(H_2), \mathrm{tr})$ is a morphism in $\mathbf{Q}$ iff $\psi = \varphi^*$. Therefore, if φ is a CPTP map, i.e. a quantum channel in the Schrödinger picture, then ψ is the corresponding channel in the Heisenberg picture, which is, in fact, an NCPU (normal completely positive unital) map. The linear logic (i.e. $*$-autonomous) duality $(-)^\perp$ in $\mathbf{Q}$ coincides with the Heisenberg-Schrödinger duality in this situation.

Schrödinger Picture	**Q**	LL_+
System description	$T(H_P)$	P
Quantum composition	$T(H_P) \hat{\otimes} T(H_R)$ $\cong$ $T(H_P \overset{2}{\otimes} H_R)$	$P \otimes R$
Classical composition	$T(H_P) \overset{1}{\oplus} T(H_R)$	$P \oplus R$

TABLE I
SCHRÖDINGER PICTURE AND POSITIVE LOGICAL POLARITY.

Heisenberg Picture	**Q**	LL_-
System description	$B(H_N)$	N
Quantum composition	$B(H_N) \overline{\otimes} B(H_M)$ $\cong$ $B(H_N \overset{2}{\otimes} H_M)$	$N \parr M$
Classical composition	$B(H_N) \overset{\infty}{\oplus} B(H_M)$	$N \& M$

TABLE II
HEISENBERG PICTURE AND NEGATIVE LOGICAL POLARITY.

Proposition IV.1. *In* **Q***, the monoidal product* $(T(H_1), B(H_1), \mathrm{tr}) \otimes (T(H_2), B(H_2), \mathrm{tr})$ *is the object* $(T(H_1) \hat{\otimes} T(H_2), B(H_1) \overline{\otimes} B(H_2), \mathrm{tr}')$*, where* $B(H_1) \overline{\otimes} B(H_2)$ *is the* spatial *tensor product, which is isomorphic to* $B(H_1 \otimes H_2)$*. Because of this isomorphism and the isomorphism* $T(H_1) \hat{\otimes} T(H_2) \cong T(H_1 \otimes H_2)$*,* tr' *can be defined via the canonical dual pairing* $T(H_1 \otimes H_2)^* \cong B(H_1 \otimes H_2)$ *of the Heisenberg-Schrödinger duality.*

The above proposition shows how to recover the spatial tensor product (composition in the Heisenberg picture) from the completely projective one (composition in the Schrödinger picture) via the Chu construction (coming from the semantics of LL). Let us explain how we can understand this situation using ideas from Polarised Linear Logic (see Tables I and II). We use P, R to range over atomic formulas with positive logical polarity, to which we associate Hilbert spaces H_P, H_R and operator spaces $T(H_P)$ and $T(H_R)$. We use N, M to range over atomic formulas with negative logical polarity, to which we associate Hilbert spaces H_N and H_M, and von Neumann algebras $B(H_N)$ and $B(H_M)$. The **Q**-columns depict the first component of the objects in **Q** corresponding to formulas. By *"Quantum composition"* we mean the space-wise composition of quantum systems, where both quantum and classical (i.e. non-quantum) interactions are possible between the two subsystems. By *"Classical composition"* we mean space-wise composition where only classical interactions are possible between the two subsystems. In order to understand intuitively why that is the case, observe that if we take the idea of classical composition to the extreme, we have that $\ell^\infty(X) \cong \bigoplus_X^\infty \mathbb{C}$ is a *commutative* von Neumann algebra, which is well-known to represent classical information in the Heisenberg picture. Its predual $\ell^1(X) \cong \bigoplus_X^1 \mathbb{C}$ can be used to represent classical information in the Schrödinger picture.

In fact, since von Neumann algebras are closed under ℓ^∞ direct sums, and their preduals under ℓ^1 direct sums, Tables I and II can be summarised (and generalised) by saying: von Neumann algebras (viewed as dual operator spaces) correspond to formulas with negative logical polarity and their preduals correspond to formulas with positive logical polarity.

V. The Quantum Switch and the Haagerup Tensor Product

The *(pure state) quantum switch* is a higher-order map that has attracted considerable interest [24]. It admits a natural definition within **OS** as a complete contraction $\mathrm{qsw}\colon B(H) \hat{\otimes} B(H) \to B(\mathbb{C}^2 \overset{2}{\otimes} H)$ determined by $f \otimes g \mapsto (|0\rangle\langle 0| \otimes (fg)) + (|1\rangle\langle 1| \otimes (gf))$ and it appears to make essential use of superposition (the "+" in its definition). The order of application of f and g is not sequential. In general, the sequentiality[1] of operator space maps can be determined via the *Haagerup tensor product* $\overset{h}{\otimes}$ [22]. This monoidal product is nonsymmetric, and satisfies the following property: there is a complete contraction $\iota : X \hat{\otimes} Y \to X \overset{h}{\otimes} Y$ such that for each complete contraction $\varphi : X \hat{\otimes} Y \to B(H)$, we have that φ factorizes via ι if and only if it can be written as $\varphi(x \otimes y) = \psi_1(x)\psi_2(y)$ for complete contractions $\psi_1 : X_1 \to B(K, H)$ and $\psi_2 : X_2 \to B(H, K)$, i.e. if it exhibits a sequential order. We showed that the pure state quantum switch does not factorize via ι, confirming that it cannot be decomposed in a reasonable sequential way. This suggests that **OS** and the Haagerup tensor give an interesting model for studying similar higher-order phenomena that make use of superposition in an essential way. In the full paper we also discuss the relevance of the Haagerup tensor to *BV logic*, an extension of MLL (Multiplicative Linear Logic) with another binary non-commutative connective called *seq*.

VI. Related Work

In follow-up work to this paper, and building upon the results established here, Li and Zamdzhiev construct a model of MALL (the exponential-free fragment of linear logic) where the linear logic duality coincides with the Heisenberg-Schrödinger duality on polarised formulas/proofs and where the relevant first-order homsets contain precisely the quantum operations (i.e. quantum channels) for *finite-dimensional* quantum theory only [25]. In contrast, the present work focuses on *infinite-dimensional* quantum theory and we recover a model of (full) linear logic including the exponentials, but in the present work, the relevant homsets contain the quantum operations, but do not coincide precisely with them. In other words, the embedding of the quantum operations/channels is not full. Recovering a fully faithful embedding in infinite-dimensions while having a model of (full) linear logic is left for future work. Both the present paper and [25] are submitted for talks to this workshop. The two submissions (including companion papers and appendices) span more than 100 pages, so there is more than enough material to present during the workshop. Should both be accepted for talks, the presenters will synchronise with each other in order to minimise overlap.

[1] More precisely, the existence of certain kinds of multilinear decompositions of maps [16].

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