Quantum Coherence Spaces Revisited: A von Neumann (Co)Algebraic Approach

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We describe a categorical model of MALL (Multiplicative Additive Linear Logic) inspired by the Heisenberg-Schrödinger duality of finite-dimensional quantum theory. Proofs of formulas with positive logical polarity correspond to CPTP (completely positive trace-preserving) maps in our model, i.e. the quantum operations in the Schrödinger picture, whereas proofs of formulas with negative logical polarity correspond to CPU (completely positive unital) maps, i.e. the quantum operations in the Heisenberg picture. The mathematical development is based on noncommutative geometry and finite-dimensional von Neumann (co)algebras, which can be defined as special kinds of (co)monoid objects internal to the category of finite-dimensional operator spaces. The full version is accepted to FoSSaCS 2026, see [LZ26] for the extended version with appendices.

1 INTRODUCTION AND CONTEXT

Linear Logic (LL) was discovered by Girard while studying the mathematical model of *coherence spaces* [Gir86; Gir87]. Later, *Probabilistic Coherence Spaces* (PCSs) [DE11] were described and they give a model of LL which is suitable for modeling discrete probability. A natural next step in the development of these models is to consider *quantum coherence spaces* (QCSs) with the intention of taking the semantic study of LL to quantum theory. Girard already proposed models of QCSs [Gir04; Gir07a; Gir07b] and they were later studied in [Bar10], but this approach has already been criticised by Selinger [Sel04] as being inappropriate for quantum theory. One of the main problems with this proposal is that the notion of morphism between QCSs does not appear to correspond to a *completely positive map* (see also [Bar10, p. 2]). This is problematic from the point of view of quantum theory, because *quantum operations* (also known as *quantum channels*) are modelled mathematically as certain kinds of completely positive maps.

In this work we set out to paint a new picture of what a quantum coherence space ought to be. We consider finite-dimensional quantum theory and we describe a model of MALL based on the following natural ideas: (1) proofs $P \vdash R$ of formulas with *positive* logical polarities should admit an interpretation as CPTP (completely positive trace-preserving) maps, i.e. as the quantum operations in the *Schrödinger picture* of quantum theory; (2) proofs $M \vdash N$ of formulas with *negative* logical polarities should admit an interpretation as CPU (completely positive unital) maps, i.e. as the quantum operations in the *Heisenberg picture* of quantum theory; (3) the LL duality should coincide with the Heisenberg-Schrödinger duality of quantum theory on polarised formulas. These desiderata naturally lead to the theory of operator spaces [ER00; BL04; Pis03] which has excellent categorical properties [LZ24] and which can be used to construct a model of (full) LL whose duality is compatible with the Heisenberg-Schrödinger duality on the level of objects/formulas [LZ25]. However, in [LZ25], the morphisms/proofs do *not* correspond to the quantum operations in either picture. We address this problem, (which is suggested for future work in [LZ25]) for *finite-dimensional* (f.d.) quantum theory and MALL.

Other works related to quantum theory and gluing and orthogonality include [SK22] which describes a model of MALL and [TA24] which describes a model of LL. Our model exhibits two main differences: first, all of our polarised formulas have physical interpretations and a mathematical formulation based on mathematical physics; second, complete positivity is not assumed a priori, but it is *derived* only where necessary, i.e. for *mixed state* quantum computation. Note that [SK22] is a follow-up work to the MLL model of [KU19] which is also based on gluing and orthogonality. Our model also has the advantage that it can also be used to reason about *pure state* quantum computation, where complete positivity obviously makes no sense.

2 BACKGROUND

We work in the setting of finite dimensional (f.d.) *operator spaces*, which are widely seen as the "non-commutative" or "quantised" generalisation of Banach spaces. An operator space is a vector space X equipped with additional norms on the vector spaces $\mathbb{M}_n(X)$, consisting of the $n \times n$ matrices with entries in X, that satisfy some conditions.

Definition 1 (Operator Space [Pis03, pp. 34–35]). *A* f.d. operator space *is a f.d. complex vector space X equipped with a sequence of norms $\| - \|_n : \mathbb{M}_n(X) \to [0, \infty)$, one for each $n \in \mathbb{N}$, such that: $\|x \oplus y\|_{m+n} = \max\{\|x\|_m, \|y\|_n\}$ and $\|\alpha x \beta\|_m \leq \|\alpha\|\|x\|_m\|\beta\|$, for $n, m \in \mathbb{N}$, $x \in \mathbb{M}_m(X)$, $y \in \mathbb{M}_n(X)$, $\alpha, \beta \in M_m$. A sequence of norms on an f.d. vector space X that satisfies the above criteria is called an* operator space structure *(o.s.s.) on X. We write $M_n(X)$ for the normed space $(\mathbb{M}_n(X), \| \cdot \|_n)$.*

Every f.d. operator space X is a Banach space w.r.t. the ground norm $\|-\|_1$ for which we simply write $\|-\|$. In the above definition, $\alpha, \beta \in M_n$ are scalar $n \times n$ matrices and the space $M_n = (\mathbb{M}_n(\mathbb{C}), \|-\|_{\text{op}})$ is equipped with the usual operator norm. In fact, M_n is canonically also an operator space with matrix norms determined via the linear isomorphism $\mathbb{M}_m(M_n) \cong M_{nm}$ and the operator norm on the latter space.

A linear map between operator spaces $\varphi : X \to Y$ induces for each $n \in \mathbb{N}$ a linear map $\varphi_n : M_n(X) \to M_n(Y)$ via componentwise application. We say that φ is *completely bounded* if $\|\varphi\|_{\text{cb}} \triangleq \sup \{\|\varphi_n\| \ : \ n \in \mathbb{N}\} < \infty$, where $\|\varphi_n\|$ is the standard operator norm of φ_n. The norm $\|-\|_{\text{cb}}$ is called the completely bounded (cb) norm. We say φ is a *complete contraction* if $\|\varphi\|_{\text{cb}} \leq 1$. We have $\|\varphi\| \leq \|\varphi\|_{\text{cb}}$, so a complete contraction is a contraction.

3 CONTRIBUTIONS

First, we prove relevant categorical properties of f.d. operator spaces. This allows us to provide an equivalent categorical definition of f.d. von Neumann algebras. We then dualise this definition to introduce f.d. von Neumann *coalgebras*. This, in turn, allows us to recover the Heisenberg-Schrödinger duality from a categorical perspective. Finally, we use these results to construct a category satisfying the desiderata (1)-(3) in Section 1.

3.1 Categorical properties of FdOS

We begin by introducing **FdOS**, the category of *f.d. operator spaces* and *linear completely contractive maps*, which constitutes the base upon which we build the successive results. This category has three notable monoidal structures given by: the *projective tensor* $\widehat{\otimes}$, which serves as the multiplicative conjunction; the *injective tensor* $\check{\otimes}$, which serves as the multiplicative disjunction; and the *Haagerup tensor* $\overset{h}{\otimes}$, which is outside the scope of LL, but very important in operator space theory. The category **FdOS** has a closed symmetric monoidal structure $(\mathbf{FdOS}, \mathbb{C}, \widehat{\otimes}, \mathrm{CB}(-,-))$, where $\mathrm{CB}(-,-)$ is the operator space of completely bounded maps. It also has (co)products given by the direct sum $\overset{\infty}{\oplus}$ $(\overset{1}{\oplus})$, which has underlying vector space given by the direct sum $\oplus$ of vector spaces, and o.s.s. given by the operator space generalizations of the ℓ_∞ and ℓ_1 norms from Banach space theory. The operator space dual $X^* \triangleq \mathrm{CB}(X, \mathbb{C})$ gives the interpretation of linear negation and it is consistent with the Banach/vector space dual for f.d. operator spaces.

Theorem 1. *The category* **FdOS** *has finite (co)products, it is* $*$*-autonomous with* $\mathbb{C}$ *as global dualizing object and is therefore a model of MALL.*

Example 2. The vector space $\mathbb{M}_n$ can be equipped with another important o.s.s. via the linear isomorphism $\mathbb{M}_n \cong M_n^* :: a \mapsto (b \mapsto \mathrm{tr}(ab))$ and the o.s.s. of the latter space. We write T_n for the resulting operator space. Its ground norm is precisely the trace norm $\|-\|_{\mathrm{tr}}$.

The Haagerup tensor is outside the scope of Linear Logic. However, it is compatible with the $*$-autonomous structure and it makes **FdOS** into a *BV-category (with negation)* in the sense of [BPS10]. This observation relies on the fact that the Haagerup tensor on f.d. operator spaces, is self-dual, meaning that $X^* \overset{h}{\otimes} Y^* \cong (X \overset{h}{\otimes} Y)^*$, see [Pis03, §5] for details.

3.2 von-Neumann (co)algebras

Operator spaces alone do not have sufficient structure for quantum computation. In the Heisenberg picture, we can use *von Neumann algebras (vN-algebras)* to define the relevant quantum operations (i.e. quantum channels). Every such algebra can be equipped with a canonical o.s.s., so we may think of them as operator spaces with additional structure. We show that vN-algebras may be equivalently defined as certain kinds of involutive monoid objects in $(\mathbf{FdOS}, \mathbb{C}, \overset{h}{\otimes})$. This novel categorical definition, and the self-duality of the Haagerup tensor, allow us to easily dualise this notion to formulate *von Neumann coalgebras* as certain involutive comonoid objects in $(\mathbf{FdOS}, \mathbb{C}, \overset{h}{\otimes})$, which we use for the Schrödinger picture, allowing for a definition that is independent from vN-algebras.

Proposition 1. *If A is a vN-algebra, then the o.s. dual A^* is a vN-coalgebra. Conversely, if C is a vN-coalgebra, then the o.s. dual C^* is a vN-algebra.*

Example 3. We can endow the operator space M_n with a vN-algebra structure using the usual matrix multiplication monoid structure and conjugate transpose serving as the involution. We then obtain an isomorphism of vN-coalgebras $T_n \cong M_n^*$ and also a vN-algebra isomorphism $T_n^* \cong M_n$ by duality.

It is well-known that all f.d. vN-algebras are of the form $\overset{\infty}{\oplus}_{1 \le i \le n} M_{k_i}$, modulo a vN-algebra isomorphism. We also prove a similar representation result for vN-coalgebras.

Proposition 2. *If C is a vN-coalgebra, then there is a vN-coalgebra isomorphism $C \cong \overset{1}{\oplus}_i T_{k_i}$.*

In order to capture quantum operations in the Heisenberg and Schrödinger pictures, we revisit the notion of *completely positive (CP)* map for each picture by giving categorical definitions that are Hilbert-space-free, i.e. abstract. For concrete vN-(co)algebras, i.e. the ones given by direct sums of M_n's (T_n's), these definitions of complete positivity coincide with the usual concrete definitions that are well-known. We further show that CPU maps are a natural notion of morphism between vN-algebras and that CPTP maps are a natural notion of morphism between vN-coalgebras. We can now define two subcategories of **FdOS**: the subcategory of f.d. vN-algebras and CPU maps, written **H**; and the subcategory of f.d. vN-coalgebras and CPTP maps, written **S** . The category **H** (**S**) is monoidal w.r.t. $\breve{\otimes}$ ($\widehat{\otimes}$) and has (co)products given by $\overset{\infty}{\oplus}$ ($\overset{1}{\oplus}$). These new formulations enable us to give a categorical formulation of the Heisenberg-Schrödinger duality as follows.

Theorem 4. *We have equivalences of categories* $(\cdot)^* \colon \mathbf{S} \simeq \mathbf{H}^{\mathrm{op}} \colon (\cdot)^*$ *and also* $(\cdot)^* \colon \mathbf{vNCoalg} \simeq \mathbf{vNAlg}^{\mathrm{op}} \colon (\cdot)^*$*, where the action on objects of* $(\cdot)^*$ *is defined as in Proposition 1. Moreover, these equivalences are strong monoidal (and (co)product preserving).*

The second duality is stronger mathematically, because it involves morphisms that preserve all of the structure of vN-(co)algebras, whereas the first duality is more relevant for quantum computation and it is used for the results in the following subsection.

3.3 Revisiting Quantum Coherence Spaces

Now we know that the (sub)categories $\mathbf{S} \hookrightarrow \mathbf{FdOS} \hookleftarrow \mathbf{H}$ have the right categorical structure for a model of MALL whose duality is induced by the Heisenberg-Schrödinger duality. However, neither **S**, nor **H**, is a *full* subcategory of **FdOS**. Luckily, there is a simple solution: a semantic technique of Hyland and Schalk [HS03], based on *gluing and orthogonality*, allows us to carve out a category **Q** from **FdOS**, such that we get fully faithful inclusions $\mathbf{S} \hookrightarrow \mathbf{Q} \hookleftarrow \mathbf{H}$, while preserving all the MALL structure. This is precisely what we do by defining a suitable notion of orthogonality/polarity based on ideas from [HS03]. The resulting model has an obvious resemblance to (probabilistic) coherence spaces, notably because they also can be recovered using gluing and orthogonality.

More specifically, we are interested in pairs (X, S), where X is an operator space and $S \subseteq \mathrm{Ball}(X)$ is a subset of the unit ball of X, which satisfies an additional condition.

Definition 2 (Polar). *Given an operator space* X *and a subset* $S \subseteq \mathrm{Ball}(X)$*, we define the* polar *of* S *to be* $S^\circ \triangleq \{f \in \mathrm{Ball}(X^*) \mid \forall s \in S.\ f(s) = 1\}$*. We say that* S *is* bipolar *if* $S = d^{-1}[S^{\circ\circ}]$*, where* $d \colon X \cong X^{**}$ *is the canonical isomorphism.*

Definition 3. *Let* $\mathbf{Q}$ *be the category whose objects are pairs* (X, S) *with* $X \in \mathbf{FdOS}$ *and* $S \subseteq \mathrm{Ball}(X)$ *a bipolar set, i.e.* $S = S^{\circ\circ}$*, and whose morphisms* $f : (X, S) \to (Y, R)$ *are complete contractions* $f : X \to Y$ *such that* $f[S] \subseteq R$*.*

For a vN-coalgebra C, we write $P_C = \{\rho \in C \mid \varepsilon(\rho) = 1 \text{ and } \rho \geq 0\}$ for the set of density operators, where ε indicates the counit of C. For instance, if $C = T_n$, then the counit is the trace and P_C is the set of density matrices. Now we can state our main result.

Theorem 5. *The category* $\mathbf{Q}$ *has finite (co)products, it is* $*$*-autonomous and is therefore a model of MALL. Moreover, the functor* $H \colon \mathbf{H} \to \mathbf{Q}$ *defined by* $H(A) \triangleq (A, \{1_A\})$ *and*

Schrödinger Picture	$\mathbf{S} \overset{\text{full}}{\hookrightarrow} \mathbf{Q}$	LL_+
System description	$\mathcal{C}, \mathcal{D}$	P, R
Quantum composition	$\mathcal{C} \mathbin{\widehat{\otimes}} \mathcal{D}$	$P \otimes R$
Classical composition	$\mathcal{C} \overset{1}{\oplus} \mathcal{D}$	$P \oplus R$
Quantum operation	$\mathcal{C} \xrightarrow{\text{CPTP}} \mathcal{D}$	$P \vdash R$

Heisenberg Picture	$\mathbf{H} \overset{\text{full}}{\hookrightarrow} \mathbf{Q}$	LL_-
System description	$\mathcal{A}, \mathcal{B}$	N, M
Quantum composition	$\mathcal{A} \mathbin{\check{\otimes}} \mathcal{B}$	$N \parr M$
Classical composition	$\mathcal{A} \overset{\infty}{\oplus} \mathcal{B}$	$N \mathbin{\&} M$
Quantum operation	$\mathcal{B} \xrightarrow{\text{CPU}} \mathcal{A}$	$M \vdash N$

Fig. 1. Schrödinger/Heisenberg picture and positive/negative logical polarity.

$H(f) \triangleq f$, is fully faithful, strict monoidal w.r.t $\check{\otimes}$, and it strictly preserves finite products. Furthermore, the functor $S\colon \mathbf{S} \to \mathbf{Q}$ defined by $S(C) \triangleq (C, P_C)$ and $S(f) \triangleq f$, is fully faithful, strict monoidal w.r.t $\widehat{\otimes}$, and it strictly preserves finite coproducts.

We are now justified in presenting the summary provided by Figure 1, on which we elaborate. In our model $\mathbf{Q}$, formulas in MALL admit interpretations as objects of $\mathbf{Q}$ and proofs are interpreted as morphisms of $\mathbf{Q}$. Formulas with positive (negative) logical polarities admit natural interpretations as vN-coalgebras $\mathcal{C}, \mathcal{D}$ (vN-algebras $\mathcal{A}, \mathcal{B}$) and proofs between such formulas correspond precisely to the CPTP (CPU) maps, i.e. the quantum operations in the Schrödinger (Heisenberg) picture. Moreover, this correspondence is preserved by classical composition (spacewise composition of systems where only a limited amount of classical interactions are possible) and by quantum composition (spacewise composition with the full range of quantum interactions possible, e.g. entanglement), in both pictures, whose interpretations are provided by the respective tables.

We showed that $\mathbf{Q}$ captures *mixed state* quantum computation in both pictures. In fact, $\mathbf{Q}$ also has interesting properties that are relevant to *pure state* quantum computation, where unitarity is very important. We have the following family of objects in $\mathbf{Q}$.

Lemma 6. *Let $u \in M_n$ be a unitary matrix. Then the pair $(M_n, \{u\})$ is an object of $\mathbf{Q}$.*

These can be composed using the projective tensor, as we have the following object equalities in $\mathbf{Q}$: $(M_n, \{u\}) \widehat{\otimes} (M_n, \{v\}) = (M_n \widehat{\otimes} M_n, \{u \otimes v\}^{\circ\circ}) = (M_n \widehat{\otimes} M_n, \{u \otimes v\})$ for unitary matrices $u, v \in M_n$. This enables us to reason about interesting higher-order (pure state) maps in $\mathbf{Q}$, such as the pure state *quantum switch*. This is the linear map $\mathrm{qsw}\colon M_n \widehat{\otimes} M_n \to M_{2n}$ defined by $\mathrm{qsw}(a \otimes b) \triangleq (|0\rangle\langle 0| \otimes (ab)) + (|1\rangle\langle 1| \otimes (ba))$ and it is a complete contraction (special case of [LZ25, §IV]).

Proposition 3. *Let $u, v \in \mathbb{M}_n$ be two unitary matrices. Then* $\mathrm{qsw}\colon (M_n \widehat{\otimes} M_n, \{u \otimes v\}) \to (M_{2n}, \{|0\rangle\langle 0| \otimes (uv) + |1\rangle\langle 1| \otimes (vu)\})$ *is a morphism of* $\mathbf{Q}$.

Therefore qsw may be recognised as a valid (unitary-preserving) higher-order map in $\mathbf{Q}$. The “higher-order” description of qsw is justified by the fact that in pure state computation, we think of the unitary matrices $u, v \in M_n$ as first-order (reversible) functions. One of the reasons qsw is interesting, is that it does not admit any reasonable multi-linear decomposition. In fact, this can be determined via the Haagerup tensor, which actually *characterises* maps that do admit similar multi-linear decompositions [ER00, §9.4], and this also can be done in $\mathbf{Q}$. See the full paper [LZ26] for more details.

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